Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH30620-WE01

Title: Topology V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 For a set X with topology τ, define:

$$\tau^c = \{A \in \mathcal{P}(X) \mid X \setminus A \in \tau\}; \quad \tau^k = \{A \in \mathcal{P}(X) \mid A \notin \tau\} \cup \{\emptyset, X\}.$$

(a) For $X = \{1, 2, 3\}$, $\tau = \{\emptyset, \{1\}, \{2, 3\}, X\}$, write down τ^c and τ^k. Is each of these a topology on X? If yes, just state this. If not, explain why not.

(b) For X finite and τ any topology on X, is τ^c always a topology on X? Is τ^k always a topology on X? Give a proof or counterexample for each.

(c) For Y infinite and τ any topology on Y, is τ^c always a topology on Y? Is τ^k always a topology on Y? Give a proof or counterexample for each.

Q2 (a) Write down definitions for i) the discrete topology and ii) compactness.

(b) Show that if τ is the discrete topology then the topological space (X, τ) is compact if and only if X is finite.

(c) If $\tau_1 \subseteq \tau_2$, is it true that (X, τ_1) is compact $\Longrightarrow (X, \tau_2)$ is compact? Is it true that (X, τ_2) is compact $\Longrightarrow (X, \tau_1)$ is compact? Give a proof or counterexample for each.

Q3 (a) State what it means for two topological spaces to be homotopy equivalent.

(b) Consider the lists of upper- and lower-case letters below (in the given font!).

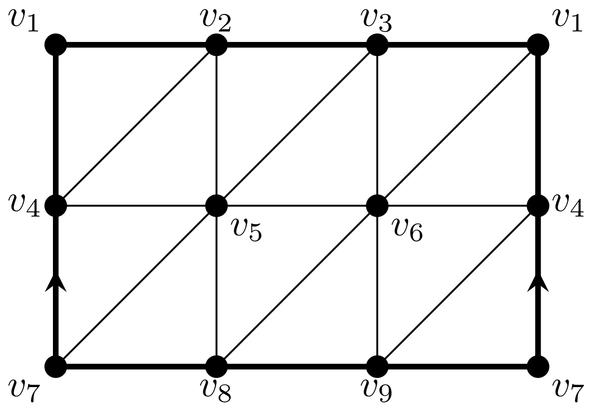
$$\begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \\ \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} \end{array}$$

Viewing each letter as a subset of $\mathbb{R}^2$ equipped with the subspace topology, partition the upper-case list, the lower-case list and the combined list, respectively, into sets of homotopy-equivalent topological spaces. In particular, identify any letters from the upper-case list which are not homotopy equivalent to their lower-case counterparts. Briefly justify your answers, including by making reference to appropriate topological invariants wherever necessary.

(c) Prove that the annulus $A = \{z \in \mathbb{C} \mid 1 \leq |z| < 2\}$ and the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ are homotopy equivalent but not homeomorphic.

Q4 (a) If K and L are finite simplicial complexes, state what it means for a map $f : K \to L$ to be a simplicial map.

(b) Let K be the 2-dimensional finite simplicial complex represented (via the identification indicated by the arrows on the left- and right-hand sides) by the diagram below which triangulates the cylinder $S^1 \times [0, 1]$, where the vertices are labelled $v_1, \ldots, v_9$.



Consider now the surjective simplicial map $f : K \to L$ determined by

$$f(v_i) = \begin{cases} w_1, & \text{if } i \in \{1, 2, 3\}, \\ w_{i-2}, & \text{if } i \in \{4, 5, 6\}, \\ w_5, & \text{if } i \in \{7, 8, 9\}, \end{cases}$$

where L is a finite simplicial complex with vertices $w_1, \ldots, w_5$.

(i) Sketch the simplicial complex L. State whether L triangulates a closed surface and, if so, identify that closed surface. Provide a brief justification for each part of your answer.

(ii) Compute the fundamental groups $\pi_1(K, v_1)$ and $\pi_1(L, w_1)$.

(iii) Deduce that the homomorphism $f_* : \pi_1(K, v_1) \to \pi_1(L, w_1)$ induced by f is surjective, but not an isomorphism.

SECTION B

Q5 (a) Give a definition of connectedness for a topological space. Let $Y = \{0, 1\}$ have the discrete topology. Show that a topological space X is connected if and only if any continuous function $f : X \longrightarrow Y$ is constant.

(b) In a topological space X, let $\{U_i\}_{i \in I}$ be a collection of subsets, each U_i connected. Suppose that for one of these subsets, U_{i_0}, we have $U_{i_0} \cap U_i \neq \emptyset$ for all $i \in I$. Show using part (a) that $\bigcup_{i \in I} U_i$ is connected.

For $x \in \mathbb{R}$, recall that $\lfloor x \rfloor$ is the largest integer $\leq x$.
In $\mathbb{R}^2$ with the standard topology, let $A = \{(x, y) \mid y = x - \lfloor x \rfloor\}$.

(c) Draw a sketch of A. Use your definition to show that it is not connected, and specify its components.

(d) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $(x, y) \mapsto (x - \lfloor x \rfloor, y\lfloor x \rfloor)$.
Sketch the image $f(A)$, and use part (b) to show that it is connected.

(e) Consider the function f, its restriction $f|_A$ to A, and its restriction $f|_C$ to some fixed component C of A. For each, state whether the function is continuous, and whether it is a homeomorphism onto its image. (Proofs are not required.)

Q6 Recall the vector space of quaternions

$$\mathbb{H} = \langle 1, i, j, k \rangle = \{q = a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

We give $\mathbb{H}$ a non-commutative multiplication defined by

$$ij = k,\ ji = -k,\ jk = i,\ kj = -i,\ ki = j,\ ik = -j,$$

extended by the usual distributive laws. Then $\mathbb{H} \setminus \{0\}$ with this multiplication is a group. We define

$$\bar{q} = a - bi - cj - dk \ \text{ and } \ |q| = \sqrt{a^2 + b^2 + c^2 + d^2},$$

and note the following three facts:

$$\overline{pq} = \bar{q}\bar{p}, \quad |pq| = |p||q|, \quad \text{ and } \quad q\bar{q} = |q|^2 = \bar{q}q.$$

[*Hint*: For this question you do **not** need to do any co-ordinate-wise multiplication. Use the three facts instead.]

(a) Show that $S^3 = \{q \in \mathbb{H} \mid |q| = 1\}$ is a subgroup of $\mathbb{H} \setminus \{0\}$.

(b) What is the usual topology on $\mathbb{H}$? Explain briefly why S^3 with the induced (subspace) topology is a topological group, and show that it is compact.

For any square matrix of quaternions $A = (a_{ij}) \in M_n(\mathbb{H})$, we define A^* as the conjugate transpose (that is, the ij^{th} entry of A^* is $\overline{a_{ji}}$) and note that $(AB)^* = B^*A^*$. Let

$$\mathrm{Sp}(n) = \{A \in M_n(\mathbb{H}) \mid AA^* = I = A^*A\}.$$

(c) Explain briefly why $\mathrm{Sp}(1) = S^3$. What topology should we use on $\mathrm{Sp}(n)$?

(d) Let $S^7 = \{\binom{p}{q} \in \mathbb{H}^2 \mid |q|^2 + |p|^2 = 1\}$. If $\binom{p}{q} \in S^7$, with $q \neq 0$, then show that

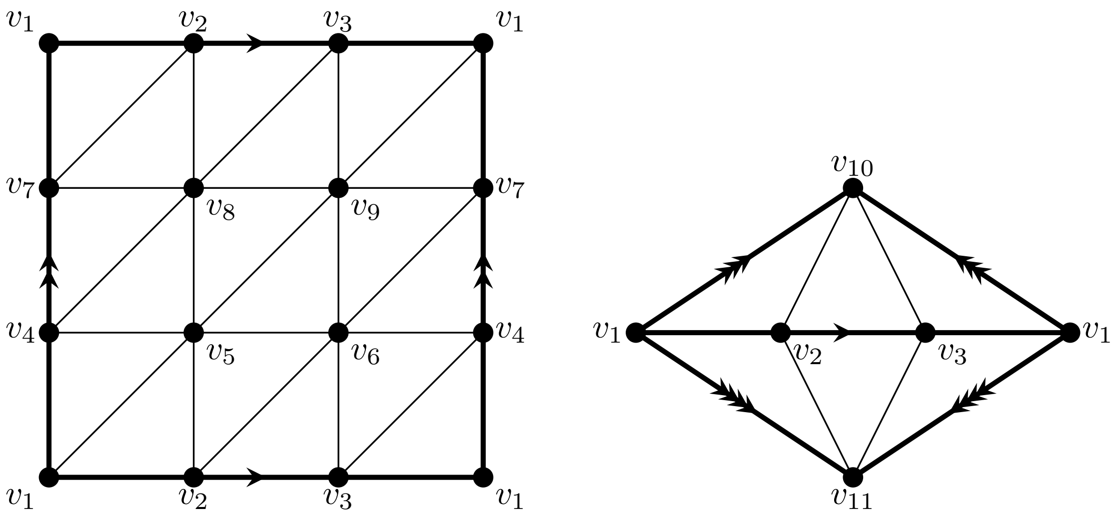
$$A = \begin{pmatrix} p & -\bar{q} \\ q & q\bar{p}q^{-1} \end{pmatrix}$$

is an element of $\mathrm{Sp}(2)$.

(e) We define a map $\bullet\colon \mathrm{Sp}(2) \times S^7 \longrightarrow S^7$ by $(A, \binom{x}{y}) \mapsto A\binom{x}{y}$.
Check that $A\binom{x}{y}$ is indeed in S^7.

(f) In fact, $\bullet$ is an action of $\mathrm{Sp}(2)$ on S^7. Show that this action is transitive.
What is the orbit space $S^7/\,\mathrm{Sp}(2)$?

Q7 Suppose that X is the connected, two-dimensional, finite simplicial complex given by $X = K \cup L$, where K and L are the connected, two-dimensional, finite simplicial complexes represented by the identification diagrams below and where the intersection $K \cap L$ is the 1-dimensional simplicial complex (triangle) common to both K and L with vertices v_1, v_2, v_3. Note that the vertices of X are labelled by $v_1, \ldots, v_{11}$, and that K and L triangulate the torus and sphere, respectively.



K L

(a) Prove or disprove the statement that X is homeomorphic to a closed surface. Briefly justify any assertions you make.

(b) Compute the Euler characteristic of X, justifying any assertions you make.

(c) Compute the fundamental group $\pi_1(X)$.

[*You may assume knowledge of the fundamental group of the circle* S^1 *and of any contractible space, if necessary, but you should present as part of your answer a calculation of the fundamental group of any other space you use.*]

Q8 (a) State the Classification Theorem for Closed Surfaces.

(b) Let Y be the surface with boundary obtained by removing the interiors of three pairwise-disjoint closed discs from a torus T.

(i) Compute the Euler characteristic of Y.

(ii) A closed surface is constructed by taking $2k$ copies of the space Y and identifying pairs of boundary circles in such a way that the final space is connected and without boundary. For each $k \in \mathbb{N}$, how many different closed surfaces (up to homeomorphism) can be constructed in this way? Justify your answer, giving explicit descriptions of the surfaces which can be constructed.