EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3071-WE01

Title:
Decision Theory III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 A certain individual may either have a particular disease, D, or not have the disease, $\overline{D}$. It is considered that the probability that the individual has the disease is 0.15.

There are two options, either treat the disease, T, or don't treat, $\overline{T}$.

Ignoring treatment costs, the utility of administering the treatment to an individual, given that they have the disease, is 10. The utility of not applying the treatment to an individual with the disease is 0. If the individual does not have the disease then the utility is 10 whether or not the treatment is administered.

The utility of the treatment cost is -2, whether the individual has the disease or not (so, for example, the overall utility of administering the treatment to an individual with the disease is 8).

Before deciding whether to give the treatment, there is an option of administering one, or two, diagnostic tests. Each test costs c utility units, for some positive value c. Each test can give a positive or a negative indication for the disease. Each tests is 75% reliable, meaning that the chance of the test giving a positive response given that the individual has the disease is 0.75 and the chance of the test giving a negative response is 0.75 given no disease. The response to each test is independent for each disease condition. The number of tests to be administered must be chosen before any test outcomes have been seen.

Find the best decision procedure, i.e. (i) the optimal number of tests to administer, (ii) the choice of whether to treat for each test outcome, and (iii) the overall utility of the best decision procedure, for each value of c.

Q2 (a) Explain what it means for two attributes of a utility function to be mutually utility independent.

(b) In a certain decision problem, suppose that the reward for the decision involves two attributes V and W, each scaled so that $0 \leq V \leq 1$, $0 \leq W \leq 1$. Denote the reward in which $V = v, W = w$ as (v, w).

Suppose that you judge the two attributes V, W to be mutually utility independent and that you judge the marginal utility of V to be linear, namely $U(V, 0) = V$.

Suppose that you also express the following isopreference curve as

$$w + v^2 = 1$$

Suppose also that you are indifferent between an outcome with attribute values (0.5,0.5) and an outcome with attribute values (0.4,0.7).

Evaluate your utility function for rewards as a function of v and w. (Any properties of utility that you require should be stated clearly but need not be proved.)

(c) Interpret what the utility function reveals about your combined attitude towards the two attributes.

Q3 (a) Consider the decision table with the utilities given below, with choice between five actions $(a_1, \ldots, a_5)$ while there are four unknown states $(\theta_1, \ldots, \theta_4)$. Uncertainty about the real state is reflected by the probabilities $P(\theta_i) = 0.25$ for $i \in \{1, 2, 3, 4\}$.

Determine the optimal decisions according to each of the following optimality criteria: (1) Maximum expected utility; (2) Maximisation of minimum utility; (3) Minimisation of maximum regret.

	θ_1	θ_2	θ_3	θ_4
a_1	9	3	6	2
a_2	4	6	6	4
a_3	1	2	8	3
a_4	4	6	6	6
a_5	3	7	2	1

(b) The individual utilities of 5 people (A-E) for 4 options (a-d) are given in the table below.

	option			
person	a	b	c	d
A	0	1	2	4
B	0	5	1	3
C	-2	2	4	6
D	1	6	3	1.5
E	0	4	4	2

Apply Harsanyi's method to combine these individual utilities into group utilities, and hence determine the group preference order.

One of the two axioms satisfied by Harsanyi's method is Anonymity; explain briefly which step(s) in the method ensure that Anonymity is satisfied.

Q4 (a) Consider the following pairwise comparison procedure to combine preference rankings of $m \geq 3$ people over $k \geq 3$ options:

For any pair of options, the group prefers one option over the other option if and only if all m people hold this preference; else the group is indifferent between these two options.

For each of the axioms in Arrow's Impossibility Theorem, explain in detail whether or not it is satisfied by this procedure. For each axiom which is not satisfied, illustrate this via an example.

(b) In approval voting, show that, if one prefers option a over option b, one would never benefit from approving option b but not approving option a.

Explain briefly whether or not approval voting might lead to strategic thinking.

SECTION B

Q5 (a) State the definition of a utility function on a set of rewards. Explain the importance of utility functions for problems of decision making under uncertainty.

(b) Suppose that you wish to specify a utility function on a set of rewards $r_1, ..., r_n$. You choose two of the rewards r_i and r_j, where you have a strict preference for r_i over r_j and you specify $U(r_i) = 1, U(r_j) = 0$. You complete your utility specification to be consistent with these assessments. Call this utility function $U_1(.)$.

You now start again and specify $U(r_i) = 10, U(r_j) = -10$. You complete your utility specification to be consistent with these assessments. Call this utility function $U_2(.)$.

Derive the relationship between U_1 and U_2.

State the general theorem for which this is a special case. Explain the relevance of this result to the use of utility functions in decision making.

(c) Suppose that two individuals, A and B, each have the same utility function for positive amounts of money, $U(X) = \sqrt{X}$, where X is the total fortune of the individual in pounds.

Discuss the risk attitude of these individuals.

Suppose that A has a current fortune of 2000 pounds and has a ticket which will pay 1000 pounds with probability 0.5 and nothing with probability 0.5. Suppose B has a current fortune of 4000 pounds. Show that there are buying prices, c, for which B would be prepared to buy the ticket from A and A would be prepared to sell the ticket to B.

Q6 We wish to estimate the parameter $\lambda > 0$ of a Poisson distribution.

The prior distribution for λ is an exponential distribution, with parameter $\beta > 0$.

The loss function, for an estimate d of parameter λ is

$$L(\lambda, d) = (d - \lambda)^2$$

[The Poisson probability function, for parameter λ, is $f(k) = e^{-\lambda}\lambda^k/k!, k = 0, 1, 2, ...$

The exponential distribution with parameter $\beta > 0$ has density $f(x) = \beta e^{-\beta x} \qquad (x > 0)$.

The Gamma distribution, parameters $\alpha, \beta > 0$, has probability density function given by $p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad (x > 0)$.]

(a) Find the Bayes rule and the Bayes risk for an immediate choice of d.

(b) Suppose that, before choosing d, we observe a sample of n independent observations, $\underline{X} = (X_1, ..., X_n)$ from the Poisson distribution.

Show that the posterior distribution for λ, if we observe $X_i = x_i, i = 1, ..., n$, is a Gamma distribution.

(c) Deduce the Bayes rule and Bayes risk after having observed these sample values.

(d) Find the sampling risk profile, for a given sample size n, namely the expected loss of taking the sample and then applying the Bayes rule, for each value of λ.

(e) Find the Bayes risk of the sampling procedure, for a given sample size n.

(f) Suppose that, for each value of λ, the cost of a sample $X_i = x_i, i = 1, ..., n$ is $c\lambda(x_1 + ... + x_n)$, for some constant c. Find the optimal choice of sample size, n.

Q7 Two people, U and V, agree to solve a joint decision problem via bargaining. They have five options, A, B, C, D, E, for which their individual utilities are as follows

	A	B	C	D	E
U	1	3	4	7	7
V	8	8	6	4	1

They decide that, if they fail to reach agreement, they will settle for a status quo option, for which they both have utility 1.

(a) Sketch the feasible region and specify the Pareto boundary for this problem.

(b) Compute the Nash Point for this problem using the definition of the Nash point. Specify the bargain corresponding to the Nash Point.

(c) Explain how the Nash Point can be derived geometrically, based directly on the Nash axioms. You must explain the use of the Nash axioms in this derivation in detail.

(d) Derive the Equitable Distribution Point for this problem and specify the corresponding bargain.

(e) Suppose that one further option becomes available, for which U has utility 8 and V has utility 2. Without calculations, explain any effects this new option has on the Nash Point and on the Equitable Distribution Point.

Q8 Consider the following pay-off table for a two-person zero-sum game, where R chooses $R1$ or $R2$, and C chooses $C1, C2, C3$ or $C4$. The pay-offs to R are as follows

	$C1$	$C2$	$C3$	$C4$
$R1$	2	3	7	5
$R2$	7	5	4	6

The pay-off to C is minus the pay-off to R.

(a) Identify all dominated strategies and explain briefly why these can be neglected for the analysis of this game.

(b) Find the minimax strategies for R and C and the value of this game.

(c) Two-person *constant-sum* games are games where the sum of the winnings of both players is the same constant value, say c, for all combinations of strategies. Such a situation might, for example, occur if the players receive a bonus for actually taking part in a zero-sum game. Explain whether or not the value of c is relevant for the minimax strategies for the players.

(d) For two-person games where the sum of the winnings of both players is not constant, explain briefly any problems that may occur when players play their minimax strategies. Include an example of such a game in your explanation.