EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
|---|---|---|
| May/June | 2024 | MATH30820-WE01 |

**Title:**

# Operations Research V

| | | |
|---|---|---|
| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | Yes | Models Permitted: Casio FX83 series or FX85 series. |

| | |
|---|---|
| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

| Revision: | |
|---|---|

ED01/2024

## SECTION A

**Q1** Find the optimal value of the following linear program. State all feasible values of $x_1$, $x_2$, and $x_3$ for which the optimal value is attained.

$$\begin{array}{rrcl} \max & x_1 - 2x_2 + x_3 & & \\ \text{subject to} & -x_1 + x_2 + 3x_3 & = & -1 \\ & x_1 + x_2 - 2x_3 & \leq & 4 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 0 \\ & x_3 & \geq & 0 \end{array}$$

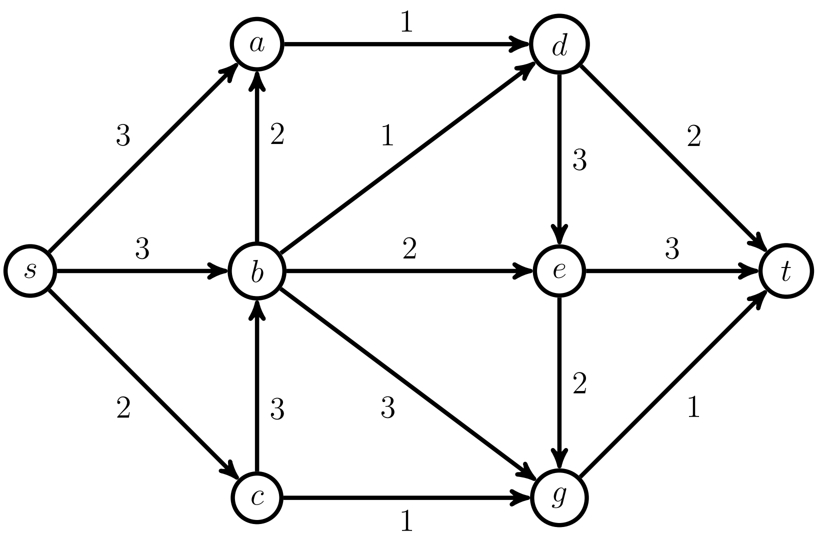
**Q2** Show that, for every linear programming problem in canonical form, if there is a feasible solution then there is a basic feasible solution.

**Q3** Consider a transportation problem with costs per unit, supplies, and demands, respectively given by:

$$[c_{ij}] = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 1 \\ 2 & 3 & 3 \\ 2 & 1 & 2 \end{bmatrix} \qquad [a_i] = \begin{bmatrix} 40 \\ 20 \\ 20 \\ 40 \end{bmatrix} \qquad [b_j] = \begin{bmatrix} 50 & 50 & 20 \end{bmatrix}$$

Find all optimal transportation schemes. If your solution is not unique, suggest a change to the cost matrix, involving as few entries as possible, that would make the solution unique. If your solution is unique, suggest a change to the cost matrix, involving as few entries as possible, that would make the solution no longer unique.

**Q4** Consider the following flow network, with capacities indicated on each arc:




Use the Ford-Fulkerson labelling algorithm to identify a maximum flow from the source $s$ to the terminus $t$, along with a minimal cut.

## SECTION B

**Q5** Ms Bee needs to decide each morning whether to stay in the same area for foraging, or whether to explore for a new location. The location she visits can be in a poor ($i = 1$) or rich ($i = 2$) state. Transition probabilities are as follows if she stays in the same area (say action $A$), where $i$ denotes yesterday's state, and $j$ denotes today's state:

$$P(j = 1 \mid i = 1) = 1 \qquad P(j = 2 \mid i = 1) = 0$$
$$P(j = 1 \mid i = 2) = 0.2 \qquad P(j = 2 \mid i = 2) = 0.8$$

If she decides to explore (say action $B$), there is a 50% chance that she ends up at a rich location, regardless of her starting location.

Each day, Ms Bee gets a reward of 2 units if that day's state is rich, and a reward of only 1 otherwise. However, exploring costs Ms Bee 1 unit. There is no cost for staying in the same location. Yesterday, Ms Bee's location was rich.

**5.1** Find a policy that maximizes Ms Bee's long-run average profit.

**5.2** As autumn draws near, exploring now costs Ms Bee two units. Is still worthwhile for Ms Bee to consider exploring?

**Q6** Consider the following linear programming problem:

$$\begin{aligned} \max \quad & x_1+3x_2+2x_3+4x_4+3x_5 \\ \text{subject to} \quad & x_1+2x_2+x_3+2x_4+x_5 \le 2 \\ & 2x_1+x_2+x_3+2x_4+2x_5 \le 3 \\ & 3x_1+3x_2+x_3+x_4+x_5 \le 1 \end{aligned}$$

and subject to all $x_i \geq 0$. The initial simplex table for this problem is

| $T_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $s_1$ | $s_2$ | $s_3$ | |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | $-1$ | $-3$ | $-2$ | $-4$ | $-3$ | 0 | 0 | 0 | 0 |
| $s_1$ | 1 | 2 | 1 | 2 | 1 | 1 | 0 | 0 | 2 |
| $s_2$ | 2 | 1 | 1 | 2 | 2 | 0 | 1 | 0 | 3 |
| $s_3$ | 3 | 3 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |

and the final simplex table is

| $T_*$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $s_1$ | $s_2$ | $s_3$ | |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 6 | 5 | 1 | 0 | 0 | 1 | 0 | 2 | 4 |
| $x_4$ | $-2$ | $-1$ | 0 | 1 | 0 | 1 | 0 | $-1$ | 1 |
| $s_2$ | $-4$ | $-5$ | $-1$ | 0 | 0 | 0 | 1 | $-2$ | 1 |
| $x_5$ | 5 | 4 | 1 | 0 | 1 | $-1$ | 0 | 2 | 0 |

Now consider a similar linear programming problem, with modifications as indicated in bold:

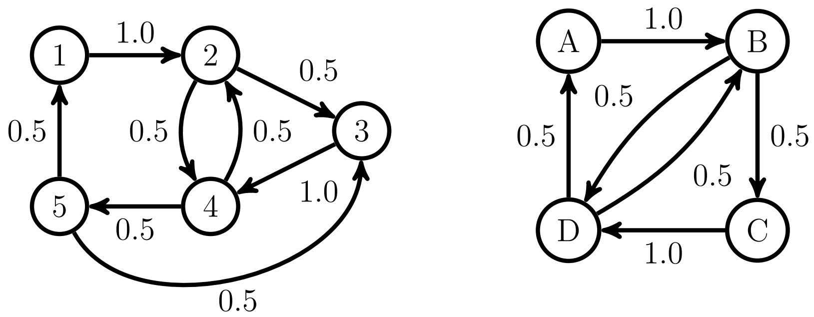
$$\max\ x_1+3x_2+2\boldsymbol{\alpha}x_3+4x_4+3\boldsymbol{\alpha}x_5$$

subject to the same constraints as before. Here, $\alpha$ is an arbitrary value in $\mathbb{R}$. For this modified problem, use post-optimal analysis to answer all of the following questions:

**6.1** For what values of $\alpha \in \mathbb{R}$ does the basis $\{x_4, s_2, x_5\}$ remain an optimal basic feasible solution? In those cases where the basis $\{x_4, s_2, x_5\}$ gives an optimal basic feasible solution, what is the optimal value of the objective function as a function of $\alpha$?

**6.2** Find the optimal solution of the modified problem for $\alpha = 2$.

**Q7** Ms Chipmunk is trying to find the most effective method to grind for experience points in her favourite computer game, 'Galactic Corn Simulator MMMXXI'. In the game, players can complete activities across the galaxy. Each activity takes 10 minutes, and has a cooldown timer, so the same activity cannot be played repeatedly in immediate succession. Ms Chipmunk's goal is to find the most profitable sequence of activities, accounting for any randomness in the game. She has identified two promising strategies, here represented as two Markov chains, where each state is an activity:




These strategies are based on disjoint sets of activities, $\{1, 2, 3, 4, 5\}$ for the strategy on the left, and $\{A, B, C, D\}$ for the one on the right. Experience points per activity are:

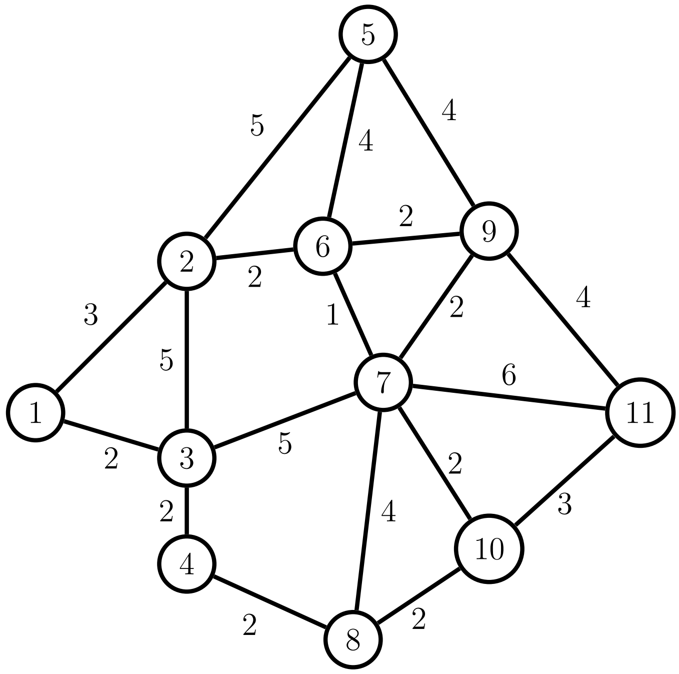
| 1 | 2 | 3 | 4 | 5 | A | B | C | D |
|---|---|---|---|---|---|---|---|---|
| 10 | 15 | 15 | 15 | 10 | 20 | 5 | 25 | 5 |

**7.1** Which of the two strategies should Ms Chipmunk use? Explain and show all relevant calculations to support your answer.

**7.2** The game developer wants to rebalance the game so that both sets of activities $\{1, 2, 3, 4, 5\}$, and $\{A, B, C, D\}$, receive more equal attention from players. They know you have helped Ms Chipmunk and are therefore eliciting your advice. How should they adjust the experience points per activity? Again, show all relevant calculations to support your answer.

You may use any results from the lectures, as long as they are stated clearly.

**Q8** Mr Frog needs to cross the pond to visit his friend Mr Beaver. He can only jump between nodes that are connected in the graph below. Here, node 1 represents Mr Frog's current location, node 11 represents the house of Mr Beaver, and all other nodes represent suitable lotus leaves. The number on the edge between nodes represent the energy that Mr Frog needs to make the corresponding jump.




**8.1** Mr Frog does not want to arrive too tired. To help Mr Frog, use Dijkstra's algorithm to identify all paths between nodes 1 and 11 that have minimal total energy.

**8.2** Mr Frog notices a juicy dragonfly sitting on node 5 that will allow him to regain 10 units of energy. However, Mr Frog knows he only has a 50% chance of catching the dragonfly. Determine whether or not it is worthwile for Mr Frog to risk a detour to try and catch the dragonfly on his way to Mr Beaver, clearly explaining your reasoning.

**8.3** Let us go back to the situation of the first part, but assume now that each jump requires the same amount of energy, say, 3 (i.e. assume all numbers on the edges of the graph are changed to 3). How could you exploit this to simplify the algorithm for finding all paths that have minimal total energy? You do not need to implement this on the graph above, but you must clearly state which steps of the algorithm can be modified for additional efficiency, and how.