EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3091-WE01

Title:
Dynamical Systems III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 The equations of a dynamical system in Cartesian coordinates (x, y) is given by

$$\begin{aligned} \dot{x} &= (\sqrt{x^2+y^2}-1)(\sqrt{x^2+y^2}-2)x - y \\ \dot{y} &= (\sqrt{x^2+y^2}-1)(\sqrt{x^2+y^2}-2)y + x \end{aligned}$$

Rewrite these equations in polar coordinates (r, φ). Sketch the phase flow of this system in polar and Cartesian coordinates and explain your reasoning for drawing that particular flow. Please label all interesting points and regions in this flow.

Q2 A linear two-dimensional dynamical system satisfies the equation $\dot{\mathbf{x}} = A\mathbf{x}$, where the matrix A is

$$A = \begin{pmatrix} 9 & -1 \\ 5 & 7 \end{pmatrix},$$

and $\mathbf{x}$ is a two-dimensional state vector.

(a) Explicitly find a similarity transformation matrix M, which transforms A into the form

$$\begin{pmatrix} r & -s \\ s & r \end{pmatrix}$$

and use this to explicitly rewrite the original system. Explicitly integrate the new equations and then transform back that solution to the original coordinates and write the solution for $\mathbf{x}(t) = (x(t), y(t))^T$.

(b) Sketch the phase flow for this linear system in the original coordinates.

Q3 Consider the dynamical system

$$\ddot{x} - (\alpha - 1)(\alpha - x)\dot{x} - x^2 = 0$$

where $\alpha \in \mathbb{R}$ is a constant.

(a) Find α such that the system is Hamiltonian.

(b) Using Bendixon's criterion, find the values of α for which the system cannot have periodic solutions lying entirely in the region $-2 < x < 2$.

(c) Consider a triangular region T in the plane, with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$, and let all the points in T evolve according to the dynamical system. What is the rate of change of the area of T at time $t = 0$, for an arbitrary α ?

Q4 **4.1** The planar dynamical system

$$\dot{x} = F(x,y), \quad \dot{y} = G(x,y)$$

has two critical points, one at $(1,1)$, and the other at $(3,3)$. You are given that

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)\bigg|_{(1,1)} = (4,0)\,, \quad \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)\bigg|_{(3,3)} = (4,-6)\,, \quad \text{and} \quad G(x,y) = x - y$$

(i) Let $\gamma(r)$ be a a circle of radius r, centred at the origin. Compute the Poincaré indices $I(\gamma(1.5))$ and $I(\gamma(4))$.

(ii) Can we define a curve γ_1, such that $I(\gamma_1) = 1$? If so, give an example.

(iii) Can we define a curve γ_1, such that $I(\gamma_1) = 2$? If so, give an example.

4.2 Consider a different planar dynamical system. You are given that $I(\gamma_3) = 2$, for some closed curve γ_3. What is the maximum and minimum number of critical points inside γ_3 ?

SECTION B

Q5 A two-dimensional dynamical system is given by the equations

$$\dot{x} = \frac{1}{2}x^2y - \frac{y}{2},$$
$$\dot{y} = \frac{1}{2}xy^2 - \frac{9}{2}x.$$

(a) Work out all fixed points of this system and determine their nature.

(b) Work out the equations for paths $y(x)$ explicitly and write orbits which go through the points $(0, \pm 2)$.

(c) Say what are homoclinic and heteroclinic orbits. If any of these exist in this system, indicate these orbits in the flow and write equations for these orbits.

(d) Using information you got from the previous part of the question, sketch the phase flow for this system.

Q6 The autonomous dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ has function $\boldsymbol{F}(\boldsymbol{x})$ given by

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{pmatrix} 10xy + 5x^2y - \alpha x \\ Ay^2 + Bxy^2 + 8 \end{pmatrix}.$$

Answer the following questions:

(a) Which properties does the function $\boldsymbol{F}(\boldsymbol{x})$ have to satisfy in general, in order for the stable manifold theorem to be valid for this sytem? State the stable manifold theorem.

(b) Explain what is the Hamiltonian system. Fix the constants A, B, α so that the system becomes Hamiltonian and then work out the Hamiltonian of this system.

(c) What is the invariant set in the phase space? Show that $x = 0$ and $x = -2$ are invariant sets. Restrict the system to the $x = -2$ subset and integrate equation for y to find all the paths in this subset.

(d) In order to sketch the phase flow for the full system, first consider submanifold $x = 0$ and sketch the phase flow for this restricted system. What are the natures of fixed points in $x = 0$ subset? How does the nature of these points change when considering them in the full phase space? Are there any paths which flow from the region $x < -2$ to $x > -2$? Incorporate all of these facts and sketch the phase flow for the full system.

Q7 **7.1** Consider the one-dimensional dynamical system $\dot{x} = F(x, \mu)$. What are the conditions for the system to undergo a saddle-node bifurcation at (x_*, μ_*) ?

7.2 Now consider the two-dimensional dynamical system

$$\dot{x} = -\cos(y)\,\sin(x) + \mu\,\cos(x)\,\sin(y)$$
$$\dot{y} = \cos(x)\,\sin(y) + \mu\,\sin(x)\,\cos(y)$$

(i) For $\mu = 0$, the dynamical system above has critical points at $(n\pi, n'\pi)$ and at $((m+\frac{1}{2})\pi, (m'+\frac{1}{2})\pi)$, for $n, n', m, m' \in \mathbb{Z}$. Determine the types of these critical points.

(ii) Plot the phase portrait, and identify the closed orbits. It is sufficient to plot what happens around $(0, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$ and (π, π).

(iii) Allow $\mu \neq 0$, with $|\mu| \ll 1$. The positions of the critical points remain the same. What are the types of the critical points now ?

(iv) What type of bifurcation occurs upon going from $\mu = 0$ to $\mu \ll 1$? Justify your answer.

Q8 Consider the two-dimensional dynamical system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

(a) Take the system to be planar. For a non-trivial periodic orbit γ, prove that

$$\int\!\!\int_{int(\gamma)} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dxdy = 0\,,$$

where $int(\gamma)$ is the interior of γ, and show that Bendixon's criterion follows from the above.

(b) A two-dimensional dynamical system is defined not on the plane, but on the surface of a cylinder with coordinates (x, y), 2π periodic in the x direction, so that (x, y) is identified with $(x+2\pi, y)$. Let D be a domain which wraps all the way round the cylinder, but which otherwise contains no holes (for example, D might be the set of points $\{(x, y) : 0 \leq x < 2\pi,\ A < y < B\}$, for two real numbers $A < B$). Assume that conditions for Bendixon's criterion hold inside D, i.e. for simple subregions we have that

$$\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)$$

is of definite sign. Show that a closed orbit lying entirely inside D, if it exists at all, must be unique.
(*Hint: proof by contradiction.*)

(c) Now consider the system

$$\dot{x} = y$$
$$\dot{y} = -y(y^2 - 4) - \sin(x)$$

Here x is again an angle so that x is identified with $x + 2\pi$, as in part (b). Prove that a closed orbit lying entirely inside the region $\{(x, y) : y > 2/\sqrt{3}\}$, if it exists at all, must be unique, and must wrap around the cylinder.