Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH31120-WE01

Title: Quantum Mechanics V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 (a) State how the adjoint, $\hat{A}^\dagger$, of an operator $\hat{A}$ is defined through the inner product $\langle\alpha|\hat{A}|\beta\rangle$ for arbitrary states $|\alpha\rangle$ and $|\beta\rangle$.

(b) Prove that any eigenvalue of a self-adjoint operator must be real.

(c) For a one-dimensional system where states are represented by normalisable wavefunctions $\psi(x)$ with $x \in \mathbb{R}$, show that $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{d}{dx}$ are self-adjoint operators with respect to the usual inner product

$$\langle\alpha|\beta\rangle = \int_{-\infty}^{\infty} dx\, \psi_\alpha^*(x)\psi_\beta(x)\ .$$

Q2 The Hilbert space of a Quantum Mechanical system has an orthonormal basis $B = \{|a\rangle,\ |b\rangle,\ |c\rangle\}$. Consider the linear operator

$$\hat{M} = 2i\,|a\rangle\langle b| + \alpha\,|b\rangle\langle a| + \beta|c\rangle\langle c|\,,$$

where α, β are in general allowed to be complex constants.

(a) You are told that $\hat{M}$ is a physical observable. How does this fix α, and constrain β?

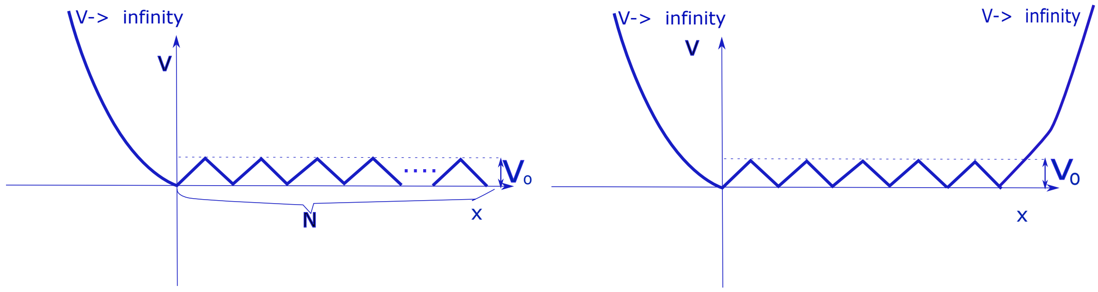
(b) Find the matrix form of $\hat{M}$ in the basis B if $\hat{M}$ is a physical observable.

(c) What are the possible outcomes of a measurement of $\hat{M}$ (leaving your answer in terms of β)?

(d) Determine the state of the system after each of the possible $\hat{M}$ measurements.

(e) Suppose you are instead told that $\hat{M}$ is not a physical observable but rather that $\hat{P} = \kappa\hat{M}^2$ is a projection operator, where κ is a constant. Use this fact to determine the only possible values of κ and β for this to be the case (such that $\hat{P}$ is not just the trivial identity operator). What are the dimensionalities of the subspaces to which the Hilbert space is projected by $\hat{P}$?

Q3 The potentials for two different one-dimensional quantum systems are shown in the figure below:



Answer the following questions:

(a) Is the energy spectrum of the particle in the first potential discrete or continuous if N is finite? If your answer depends on the energy of the particle, please explain and state which part of the spectrum is discrete and which is continuous.

(b) Analyse the spectrum of the particle in the first potential in the limit $N \to \infty$.

(c) What is the nature of the spectrum of the particle in the second potential?

(d) Sketch the qualitative form of the wave function for the semi-classical particle in the second potential, ie for the particle for which $E \gg V_0$. In particular, when sketching the wave function, indicate where the probability to find a particle is biggest and smallest, and how the frequency of the wave function changes in the region where the wave function is oscillatory.

Q4 Apply the WKB approximation to compute the energy spectrum of a quantum particle of mass $m = 1$, propagating in one dimension in the presence of a potential

$$V(x) = \frac{x^2}{2} + x\,.$$

Sketch the graph of this function, and compare the results with the spectrum of a simple harmonic osicillator with the potential $V_{SHO} = x^2/2$, explaining the origin of differences and similarities.

Hint: When evaluating integrals, you can use the expression

$$\int \sqrt{a^2 - x^2} = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right)\,.$$

SECTION B

Q5 A quantum system has only two energy eigenstates, $|1\rangle$, $|2\rangle$, corresponding to the energy eigenvalues E_1, E_2. Apart from the energy, the system is also characterized by a physical observable called parity, whose operator $\hat{\mathcal{P}}$ acts on the energy eigenstates as follows:

$$\hat{\mathcal{P}}|1\rangle = |2\rangle \ , \quad \hat{\mathcal{P}}|2\rangle = |1\rangle \ .$$

(a) Assuming that the system is initially in a positive-parity eigenstate, find the state of the system at any time.

(b) At a particular time, t, a parity measurement is made on the system.What is the probability of finding the system with positive parity?

(c) Suppose that in an attempt to track the parity we make a large number N of parity measurements throughout the time interval t. A 'survival' is when we make a series of positive parity measurements at the times $\Delta t, 2\Delta t, ..., N\Delta t = t$. What is the 'survival probability' P_N, namely the probability that we will still find the system with positive parity at time t?

(d) Use your result to find $\lim_{N\to\infty} P_N$. Does the result agree with the central assumption of quantum mechanics about the result of repeated measurements on quantum mechanical systems? If so, why? If not, why not?
[**Hint:** You may find the following useful: $\lim_{N\to\infty}(1 - a/N)^N = e^{-a/N}$]

Q6 The Hamiltonian for the simple harmonic oscillator is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \ .$$

In addition we define the annihilation and creation operators as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}\left(i\hat{p} + m\omega\hat{x}\right) \ , \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(-i\hat{p} + m\omega\hat{x}\right) \ .$$

(a) Using the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ determine the commutator $[\hat{a}, \hat{a}^\dagger]$.

(b) Use your result to express $\hat{H}$ in terms of the number operator, $\hat{n} = \hat{a}^\dagger\hat{a}$.

(c) Hence show that $\hat{a}^\dagger$ and $\hat{a}$ act on the states as raising and lowering operators of the energy respectively, namely

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ \ ; \quad a|n\rangle = \sqrt{n}|n-1\rangle \ .$$

(d) Hence or otherwise show that the matrix elements of $\hat{x}^2$ in the n-basis, that is $\langle n|\hat{x}^2|n'\rangle$, vanish unless $n = n'$ or $n = n' \pm 2$.

(e) Compute the matrix element $\langle n|\hat{x}|\ell\rangle$, and verify the completeness of the energy eigenbasis by checking explicitly that

$$\langle n|\hat{x}^2|n'\rangle = \sum_{\ell=0}^{\infty}\langle n|\hat{x}|\ell\rangle\langle\ell|\hat{x}|n'\rangle \ .$$

Q7 A quantum particle of mass m is moving in two dimensions in the presence of the potential

$$V(r) = B\rho^n , \qquad \rho^2 = x^2 + y^2$$

where B is a constant and n is an integer.

The Laplacian in polar coordinates is

$$\partial_x^2 + \partial_y^2 = \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\phi^2$$

Answer the following questions:

(a) Write the ansatz for the wave function of this quantum particle, and by separation of variables, work out the radial and angular equations. What are the solutions of the angular equation?

(b) By changing variables via $W(\rho) = \rho R(\rho)$, simplify the radial equation and explain its form.

(c) For what values of constants B and n does the system have bound states?

(d) For $n = 2$, find the general solution for the wave function of the ground state and write and explain the conditions which are used to fix the undertermined integration constants in that solution. Also find the energy of the ground state. You may use the ansatz for the radial part of the wave function $W(\rho) = Qe^{\alpha\rho^2}$.

Q8 A simple harmonic oscillator in two dimensions is perturbed by the potential

$$\hat{H}' = \alpha\bigg(\Big(\log(\hat{I} + \hat{a}_x\hat{a}_y^\dagger\Big) + \Big(\log(\hat{I} + \hat{a}_x^\dagger\hat{a}_y\Big) \bigg) ,$$

where $\hat{I}$ is an identity operator. Answer the following questions:

(a) Explicitly write down the spectrum of the unperturbed system, i.e. write down the properly normalized energy eigen-states as well as their energies. What is the degeneracy of the N-th excited level? Please explain your answer.

(b) By explicit computation find the first order correction to the energy of the ground state.

(c) By explicit computation find the first order correction to the ground state.

(d) By explicit computation find the corrections to the energy of the second excited states, as well as corrections to the states.