

FORMULA SHEET

Some vector identities:

$$\nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f\nabla \cdot \mathbf{A} \quad (1)$$

$$\nabla \times (f\mathbf{A}) = (\nabla f) \times \mathbf{A} + f\nabla \times \mathbf{A} \quad (2)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (3)$$

$$\mathbf{A} \cdot \nabla \mathbf{A} = \frac{1}{2} \nabla |\mathbf{A}|^2 - \mathbf{A} \times (\nabla \times \mathbf{A}) \quad (4)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad (5)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} \quad (6)$$

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \quad (7)$$

In cylindrical coordinates (r, θ, z) :

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{\partial f}{\partial z} \hat{\mathbf{e}}_z \quad (8)$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \quad (9)$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{\mathbf{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\mathbf{e}}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{\mathbf{e}}_z \quad (10)$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \quad (11)$$

$$\nabla^2 \mathbf{A} = \left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} \right) \hat{\mathbf{e}}_r + \left(\nabla^2 A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2} \right) \hat{\mathbf{e}}_\theta + (\nabla^2 A_z) \hat{\mathbf{e}}_z \quad (12)$$

$$\mathbf{B} \cdot \nabla \mathbf{A} = \left(\mathbf{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r} \right) \hat{\mathbf{e}}_r + \left(\mathbf{B} \cdot \nabla A_\theta + \frac{B_\theta A_r}{r} \right) \hat{\mathbf{e}}_\theta + (\mathbf{B} \cdot \nabla A_z) \hat{\mathbf{e}}_z \quad (13)$$

In spherical coordinates (r, θ, ϕ) :

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\mathbf{e}}_\phi \quad (14)$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \quad (15)$$

$$\nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right) \hat{\mathbf{e}}_r + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right) \hat{\mathbf{e}}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{\mathbf{e}}_\phi \quad (16)$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r f) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \quad (17)$$

$$\begin{aligned} \nabla^2 \mathbf{A} = & \left(\nabla^2 A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{\partial A_\phi}{\partial \phi} \right] \right) \hat{\mathbf{e}}_r \\ & + \left(\nabla^2 A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\phi}{\partial \phi} \right) \hat{\mathbf{e}}_\theta \\ & + \left(\nabla^2 A_\phi + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\theta}{\partial \phi} - \frac{A_\phi}{r^2 \sin^2 \theta} \right) \hat{\mathbf{e}}_\phi \end{aligned} \quad (18)$$

$$\begin{aligned} \mathbf{B} \cdot \nabla \mathbf{A} = & \left(\mathbf{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r} - \frac{B_\phi A_\phi}{r} \right) \hat{\mathbf{e}}_r + \left(\mathbf{B} \cdot \nabla A_\theta - \frac{B_\phi A_\phi}{r} \cot \theta + \frac{B_\theta A_r}{r} \right) \hat{\mathbf{e}}_\theta \\ & + \left(\mathbf{B} \cdot \nabla A_\phi + \frac{B_\phi A_r}{r} + \frac{B_\phi A_\theta}{r} \cot \theta \right) \hat{\mathbf{e}}_\phi \end{aligned} \quad (19)$$

Bessel functions: $u(r) = J_n(r)$ and $u(r) = Y_n(r)$ are solutions to the ODE

$$r^2 u'' + r u' + (r^2 - n^2) u = 0. \quad (20)$$

Both $J_n(r)$ and $Y_n(r) \rightarrow 0$ as $r \rightarrow \infty$; $J_n(0) = \delta_{n0}$, and $|Y_n(r)| \rightarrow \infty$ as $r \rightarrow 0$.