Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH31620-WE01

Title:
Fluid Mechanics V

Time:	3 hours	
Additional Material provided:	Formula sheet.	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2024

SECTION A

Q1 A fluid moves two-dimensionally so that its velocity, $\boldsymbol{u}$, is given by

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathrm{e}^{t}\,\widehat{\boldsymbol{e}}_x + \mathrm{e}^{-t}\,\widehat{\boldsymbol{e}}_y,$$

where $\widehat{\boldsymbol{e}}_x$ and $\widehat{\boldsymbol{e}}_y$ are the unit vectors for the Cartesian coordinates (x, y).

1.1 Define the terms *streamline*, *particle path* and *streakline*.

1.2 Obtain equations in terms of x and y alone for:

(i) the streamline through (1,1) at time $t = 0$,

(ii) the particle path for a particle released from $(1, 1)$ at time $t = 0$,

(iii) the streakline at $t = 0$ formed by particles released from $(1, 1)$ at times $t \leq 0$.

1.3 Sketch these three curves on the same diagram in the (x, y) plane.

Q2 Water in a cylindrical bucket of radius a, whose axis is vertical, is moving with velocity

$$\boldsymbol{u} = -ky\,\widehat{\boldsymbol{e}}_x + kx\,\widehat{\boldsymbol{e}}_y,$$

where k is a constant. The bucket is stationary and the water is modelled as inviscid fluid under gravity.

2.1 Confirm the flow is *steady* and *incompressible*.

2.2 Define the stream function and show that the streamlines are circles with their centre on the axis.

2.3 Use the Euler equations to find the pressure, p, at any point in the fluid at a distance r from the axis and z from the bottom of the bucket. On the free surface, which rises to $z = h$ at the edge of the bucket, the pressure is constant and you can set it to zero, $p = 0$.

Q3 Consider the incompressible unforced Navier–Stokes equations.

3.1 Nondimensionalise these equations by introducing the following dimensionless variables

$$\boldsymbol{u}' = \frac{1}{U}\boldsymbol{u}, \quad \boldsymbol{x}' = \frac{1}{L}\boldsymbol{x}, \quad t' = \frac{U}{L}t, \quad p' = \frac{1}{P}p$$

where P is a variable you should define and L and U are some typical length scale and speed. Express your dimensionless equations in term of the Reynolds number,

$$Re = \frac{UL}{\nu}.$$

3.2 Show that with constant pressure, and in the high Reynolds number limit, the momentum equation reduces to the Burgers equation when the flow takes the form $\boldsymbol{u}'(\boldsymbol{x}', t') = u'(x', t')\widehat{\boldsymbol{e}}_x$.

3.3 Are there any restrictions on the allowable forms we can take for $u'(x', 0)$ in this case?

Q4 A fluid is described by the barotropic Euler equations with body force $\boldsymbol{f} = -\boldsymbol{\nabla} U$ and enthalpy $h(\rho)$.

4.1 Show that for a time dependent potential flow that the modified Bernoulli's function

$$B(\boldsymbol{x}, t) = \frac{\partial \phi}{\partial t} + h(\rho) + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + U,$$

is conserved *everywhere*.

4.2 Show that enthalpy $h(\rho)$ is related to pressure $p = P(\rho)$ by

$$h(\rho) = \int_0^\rho \frac{P'(\sigma)}{\sigma}\, \mathrm{d}\sigma.$$

4.3 Calculate $h(\rho)$ for an adiabatic gas with $P = k\rho^\gamma$, where $\gamma, k \in \mathbb{R}$ and $\gamma > 1$ and $k > 0$.

SECTION B

Q5 A uniform stream of clear fluid with velocity $U\widehat{\boldsymbol{e}}_x$ at infinity flows past a circular cylinder of radius a. The surface of the cylinder is porous and dyed fluid is forced out with outward normal velocity $2U$ at the surface. Both fluids are inviscid and $U > 0$.

5.1 The flow is incompressible and irrotational. Show how these conditions lead to Laplace's equation for the velocity potential, ϕ.

5.2 It is proposed that the flow field can be given by the velocity potential

$$\phi = \phi_1 + \phi_2,$$

where

$$\phi_1 = U\left(1 + \frac{a^2}{r^2}\right) r\cos\theta \qquad \text{and} \qquad \phi_2 = 2aU\log r,$$

where $r = 0$ is at the centre of the circular cross-section of the cylinder.

(i) Briefly describe the flow fields that ϕ_1 and ϕ_2 represent individually.

(ii) *Verify* that the combined potential, ϕ, satisfies the boundary conditions of the full problem at large distance and on the surface of the cylinder.

5.3 Using the combined potential, show that the dyed fluid extends a distance $(1 + \sqrt{2})a$ upstream from the centre of the circle.

5.4 Sketch the streamlines of the motion, distinguishing between the dyed and undyed fluid.

5.5 By considering the downstream flow and the meaning of ϕ_2, or otherwise, determine the width of the region far downstream occupied by the dyed fluid.

Q6 Peter is sitting in a small fishing boat in the middle of a very deep Scottish lake. He watches a small-amplitude wave progressing in the positive x-direction on the surface of the lake. Ignoring the presence of the boat, the lake is modelled to have infinite depth and to be filled with water of constant density. The equation of the surface is $z = \eta(x,t)$, where z is measured vertically upwards from the undisturbed surface. The linearised dynamic boundary condition at the surface, $z = 0$, can be written

$$\frac{\partial \phi}{\partial t} + g\eta = 0,$$

where ϕ is the velocity potential and g is the acceleration due to gravity.

6.1 (i) State the partial differential equation governing ϕ in the interior of the fluid.

(ii) By considering the motion of a particle on the free surface, show that the other linearised boundary condition on ϕ at $z = 0$ is

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}.$$

(iii) State, and briefly justify, a boundary condition on ϕ as $z \to -\infty$.

6.2 Consider an ansatz for the potential of the form

$$\phi(x,z,t) = \mathrm{e}^{kz}\cos\left[k(x-ct)\right].$$

Show that the wave speed, c, is given by $c = \sqrt{g/k}$, and hence calculate the free surface height, $\eta(x,t)$.

6.3 Suppose that a second wave, with potential

$$\phi_2(x,z,t) = \mathrm{e}^{kz}\cos\left[k(x+ct)\right],$$

of the same amplitude but propagating in the opposite direction is also present. Show that the potential for the combined motion can be written in the form

$$\phi_{\text{combined}} = 2\mathrm{e}^{kz}\cos\alpha\cos\beta,$$

where you should state α and β. You may wish to use the identity

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

6.4 Peter is watching the motion of the end of his fishing line, which is submerged in the lake. He sees that it oscillates backwards and forwards in a *straight line*. He claims this must be the result of a very large creature below the surface.

Calculate (to leading order) the particle paths from the combined motion, and hence determine whether Peter has found a large creature in the lake, or whether the straight line oscillations are just the motion of the water.

Q7 Consider an initially stationary viscous fluid between two walls at $y = 0$ and $y = h$ subject to no external forces. The wall at $y = 0$ impulsively jerks into motion at $t = 0$ so that this wall moves at speed U in the x-direction for all later times. We will assume that the flow between the walls then takes the form $\boldsymbol{u} = u(y,t)\widehat{\boldsymbol{e}}_x$.

7.1 Assuming pressure remains constant ($p = p_0$), show that the incompressible Navier–Stokes equations reduce to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

for this flow and state the boundary conditions (valid for $t > 0$) and initial condition on $u(y,t)$.

7.2 Assume that $u(y,t) = f(\eta)$, where $\eta = \dfrac{y}{\sqrt{\nu t}}$, and find a similarity solution for $u(y,t)$. You may use that

$$\int \mathrm{e}^{-s^2}\,\mathrm{d}s = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(s) + c,$$

where $\mathrm{erf}(s)$ is the error function and $c \in \mathbb{R}$.

7.3 Find the long-time behaviour of this solution, i.e. $\lim_{t\to\infty} u(y,t)$. You may use that $\mathrm{erf}(s) \approx \frac{2}{\sqrt{\pi}}s + \mathcal{O}(s^2)$ for small s.

7.4 Solve the steady state Navier–Stokes equations for $u(y,t)$ with the same boundary conditions and confirm this matches your previous answer.

Q8 Lost in thoughts of maths puns, Adam walks into a lamp post. This sets the lamp post vibrating, generating sound waves. We can model the lamp post as a cylinder of radius a, and height h. Adam's head impacts the lamp post at a height d, so that in cylindrical coordinates the lamp post moves such that at $r = a$,

$$u_r(a, \theta, d, t) = U_0 \cos(\theta) \mathrm{e}^{-\mathrm{i}\omega t}.$$

At the ground ($z = 0$), $u_r = u_\theta = 0$, while the top is free to move. The lamp post therefore pivots at $z = 0$ and sways back and forth above. We will model the sound waves this creates as linear waves with $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ where

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi.$$

8.1 Express the boundary conditions above in terms of ϕ.

8.2 Assuming that ϕ takes the form $\phi(r, \theta, z, t) = \cos(\theta) R(r) Z(z) \mathrm{e}^{-\mathrm{i}\omega t}$, show that the wave equation implies that

$$Z''(z) - \lambda Z(z) = 0,$$
$$s^2 R(s)'' + s R(s)' + (s^2 - 1) R(s) = 0,$$

where s is a variable you should define.

8.3 With $\lambda = 0$, find ϕ in the region $0 \le z \le h$ that is consistent with these boundary conditions. You may use that $J_1'(t) = J_0(t) - \frac{1}{t} J_1(t)$ and $Y_1'(t) = Y_0(t) - \frac{1}{t} Y_1(t)$ where J and Y are Bessel's functions of the first and second kind.

8.4 What kind of motion of the lamp post does setting $\lambda = 0$ describe?