EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3171-WE01

Title:
Mathematical Biology

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 The population model game.

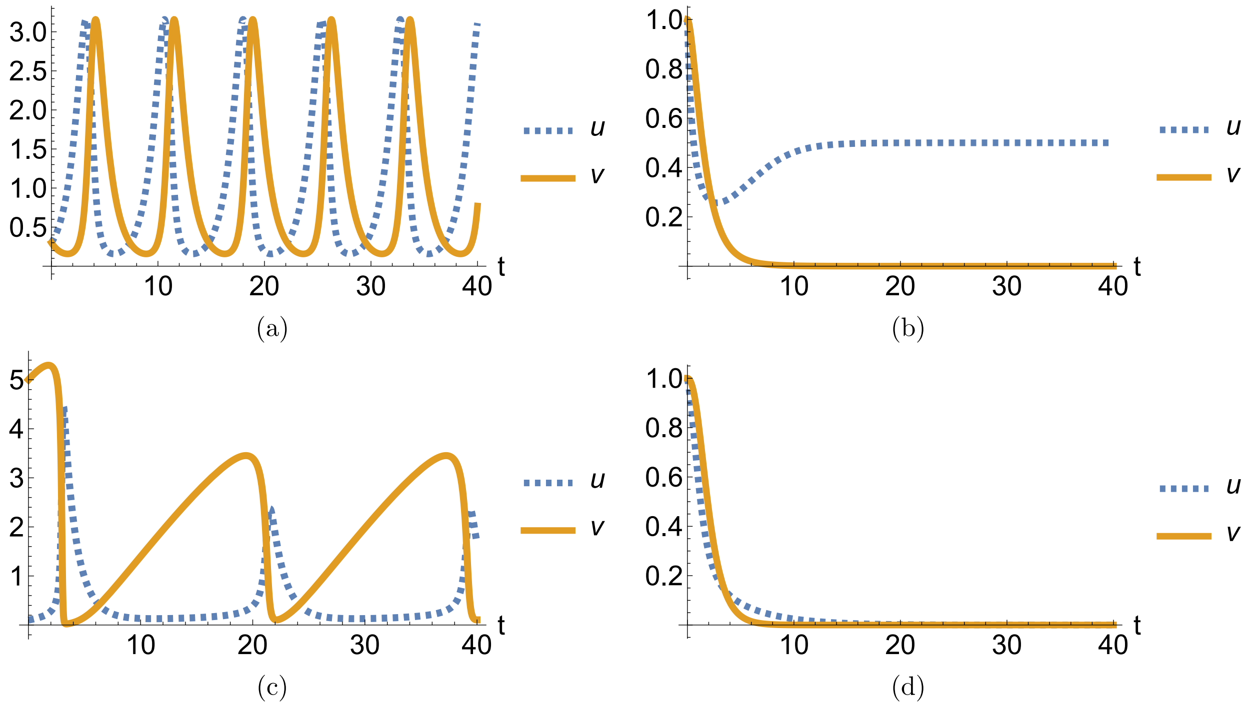

Figure 1: Plots associated with population models defined in question1

$$\text{(i)}: \quad \frac{\mathrm{d}u}{\mathrm{d}t} = u(1/2 - u^2 + v), \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -v + uv \qquad \text{(ii)}: \quad \frac{\mathrm{d}u}{\mathrm{d}t} = u(1 - v), \quad \frac{\mathrm{d}v}{\mathrm{d}t} = v(u - 1)$$

$$\text{(iii)}: \quad \frac{\mathrm{d}u}{\mathrm{d}t} = u(u^2 - 1/2 + v), \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -v + uv \qquad \text{(iv)}: \quad \frac{\mathrm{d}u}{\mathrm{d}t} = a - u - u^2 v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = b + u^2 v,$$

State the physically valid equilibria in each of the models labelled (i)-(iv), you can assume $a, b > 0$ for (iv). Match these population models to exactly one of the solution sets $(u(t), v(t))$ shown in Figure 1. You must give a reason for each match.

Q2 Collapsing microtubules. Consider a model of a microtubule embedded in a cellular matrix as an Euler beam of length L, density ρ, and bending resistivity B, deforming under an applied load N. The microtubule is placed in a visco-elastic medium which dampens the velocity of the buckling by resisting its deformation. Modelling the medium as having a viscoelastic "spring" component with constant β, and a purely elastic component with spring constant μ which resists bending, the modified beam equation for the deflection d, as a function of arclength s and time t, takes the form

$$B\frac{\partial^4 d}{\partial s^4} + N\frac{\partial^2 d}{\partial s^2} + \mu d + \beta\frac{\partial d}{\partial t} + \rho\frac{\partial^2 d}{\partial t^2} = 0,$$

with B, ρ, N, β and μ positive constants.

(a) Show, using a linear stability analysis, that the homogeneous equilibrium $d(s,t) = 0$, $\forall s$ is unstable under a critical load N_c for a given oscillatory integer) mode n when

$$N_c = \frac{Bn^2\pi^2}{L^2} + \frac{\mu L^2}{n^2\pi^2},$$

if we assume the beam is pinned: $d(0,t) = d(L,t) = 0$.

(b) Describe the relationship between the critical load N_c and its length, and comment on how the physical parameters B and μ affect this relationship (you should include their physical interpretation in your answer).

Q3 A simple Turing model. Consider the following system of reaction-diffusion equations:

$$\frac{\partial u}{\partial t} = a - u + uv + \varepsilon \frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial v}{\partial t} = b - uv + \frac{\partial^2 v}{\partial x^2},$$

with $a > 0, b > 0$ and $u, v \geq 0$ satisfying Neumann boundary conditions on the interval $x \in [0, L]$.

(a) Find the spatially homogeneous equilibrium and state any conditions on a, b for it to be feasible.

(b) Show that for this equilibrium to be stable in the absence of diffusion, we must have $b > a + \sqrt{a}$.

(c) Linearize the system around this equilibrium. What form do solutions to the linear system take? You do not need to solve the linearized equations.

(d) Explain why $\varepsilon < 1$ is necessary for this system to exhibit Turing instability.

Q4 Hyper-diffusive survival Consider a population which grows according to

$$\frac{\partial u}{\partial t} = \nabla^2 u - D\nabla^4 u + u(1-u), \tag{1}$$

on a square 2D domain $(x, y) \in \Omega = [0, L] \times [0, L]$ satisfying generalized Dirichlet conditions of the form:

$$u = 0 = \nabla^2 u \quad \text{for} \quad x \in \partial\Omega.$$

Assume that $D > 0$.

(a) Find the spatially homogeneous equilibrium of this model and state its stability in the spatially homogeneous case.

(b) State the eigenfunctions and eigenvalues of the Helmholtz equation:

$$\nabla^2 w(x, y) = -\rho w(x, y)$$

on the domain Ω. Explain why these eigenfunctions can still be used to solve the linearized version of (1).

(c) Use a linear stability analysis to show that the population will die out for all $L < L_c = \pi^2(1 + \sqrt{1 + 4D^2})$, but will persist for $L > L_c$.

SECTION B

Q5 A predator prey model. Consider the following modified variant of the Lotka-Volterra system for population densities $\bar{u}, \bar{v}$ on a one dimensional domain $x \in [0, \bar{L}]$:

$$\begin{aligned}\frac{\partial \bar{u}}{\partial \bar{t}} &= D_1 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + a\bar{u}\mathrm{e}^{-\bar{u}} - b\bar{u}\bar{v}, \\ \frac{\partial \bar{v}}{\partial \bar{t}} &= D_2 \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} - c\bar{v}\mathrm{e}^{-\bar{v}} + d\bar{u}\bar{v},\end{aligned} \tag{2}$$

where D_1, D_2, a, b, c, d are positive constants.

(a) Show that (2) can be written in the following non-dimensionalised form:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + u\mathrm{e}^{-\beta_1 u} - uv, \\ \frac{\partial v}{\partial t} &= D\frac{\partial^2 v}{\partial x^2} - \gamma v\mathrm{e}^{-\beta_2 v} + uv.\end{aligned} \tag{3}$$

State the physical interpretation of the constants D, γ and state the scaled length L of the system.

(b) Using a phase diagram (or otherwise) compare the modified growth model

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u\mathrm{e}^{-u}, \tag{4}$$

to standard exponential growth.

(c) Now consider the following purely temporal version of this system

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= u\mathrm{e}^{-u} - uv, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -v\mathrm{e}^{-v} + uv.\end{aligned}$$

Draw a phase diagram for this system featuring the equilibria, nullclines and some indicative trajectories.

Q6 Travelling waves with advection. Consider the following reaction-advection-diffusion equation for a population density $u(x,t)$

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ku\frac{\partial u}{\partial x}, \tag{5}$$

where D, k are positive constants.

(a) State the assumptions on the flux J and advective velocity required for this equation to fit the general advection-diffusion form.

(b) Show that travelling wave solutions to this equation in the form $u(z)$, $z = x - ct$ must satisfy the following equation

$$D\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + (c + ku)\frac{\mathrm{d}u}{\mathrm{d}z} = 0. \tag{6}$$

(c) Show by integrating that (6) can be written as:

$$\frac{\mathrm{d}u}{\mathrm{d}z} = -a(u - u_1)(u - u_2). \tag{7}$$

where $u_2 > u_1$. State the values of a, u_1, u_2 in terms of c, k, D.

(d) State the appropriate boundary conditions for a travelling wave which transitions between the system's two equilibria u_1, u_2 (assuming they are real). Next solve for $u(z)$ by integrating (7) ensuring the solution satisfies these boundary conditions.

Q7 A discrete Allee model. Consider the discrete-time model:

$$u_n = u_{n-1} + ru_{n-1}(1 - u_{n-1})(u_{n-1} - A), \tag{8}$$

where $0 < A < 1$ and $r > 0$. Also consider the continuous-time analogue:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru(1 - u)(u - A). \tag{9}$$

(a) Find all feasible equilibria of both models, and compute their stability in terms of r and A.

(b) Draw cobweb diagrams to illustrate what happens to initial conditions $u_0 \in (0, A)$ and $u_0 \in (A, 1)$ in the case of stable equilibria for equation (8).

(c) What unphysical behaviour does the discrete-time model exhibit for $2 > rA > 1$? What about for $rA > 2$? Illustrate the first of these with a cobweb diagram.

(d) For a fixed value of A, for what values of r does the model (8) behave qualitatively the same way as equation (9)? Besides unphysical behaviour, what else can happen in the discrete-time model which cannot happen in the continuous-time one?

Q8 Chasing resources for patterns Consider a resource-consumer model of the form,

$$\frac{\partial r}{\partial t} = D_r \nabla^2 r + 1 - r - \frac{acr}{1+r},$$
$$\frac{\partial c}{\partial t} = D_c \nabla^2 c - c + \frac{acr}{1+r},$$

where all parameters are positive and we assume that the spatial domain has no-flux boundary conditions. Here r represents the density of a resource population, and c a consumer.

(a) What do all of the terms in the system represent? In particular, what does the denominator on the interaction term mean?

(b) Find both spatially homogeneous equilibria. Classify their feasibility and stability in the absence of diffusion in terms of the parameter a.

(c) Show that neither equilibrium can undergo a Turing instability.

(d) Now consider the following variant of the model,

$$\frac{\partial r}{\partial t} = \nabla^2 r + 1 - r - \frac{acr}{1+r},$$
$$\frac{\partial c}{\partial t} = \nabla^2 c + \gamma \nabla \cdot (p \nabla r) - c + \frac{acr}{1+r},$$

with $\gamma > 0$, and the cross-diffusion Turing instability conditions derived in lectures,

$$d_1 G_v + d_4 F_u - d_2 G_u - d_3 F_v > 0,$$
$$(d_1 G_v + d_4 F_u - d_2 G_u - d_3 F_v)^2 - 4(d_1 d_2 - d_2 d_3)(F_u G_v - F_v G_u) > 0.$$

Argue why there must exist a $\gamma_c > 0$ such that this system has Turing instabilities for $\gamma > \gamma_c$. *Note: You do not need to compute γ_c exactly but you must show that it exists.*

(e) What is the interpretation of the term involving γ, noting that $\gamma > 0$?