EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3201-WE01

Title:

Geometry III

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Let H be the set of all elements $g \in Isom(\mathbb{E}^2)$ such that g has finite order.

(a) Is it true or false that H is a group (with composition as the group operation)? Justify your answer.

(b) Let H_2 be the set of all elements $g \in Isom(\mathbb{E}^2)$ such that g has order 2. Find the minimal (by inclusion) group containing H_2. Justify your answer.

Q2 Let $ABCD$ be a convex hyperbolic quadrilateral with all vertices lying inside $\mathbb{H}^2$.

(a) Show that $|AC| + |BD| \geq |AB| + |CD|$.

(b) Let P be the perimeter of $ABCD$, $P = |AB| + |BC| + |CD| + |DA|$. And let $S = |AC| + |BD|$. Find functions $f(x)$ and $g(x)$, where $f, g : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, such that

$$f(S) \leq P \leq g(S).$$

Justify your answer.

Q3 Let $\triangle ABC$ be a spherical triangle with $\angle BAC = \pi/2$ and $|AB| = |BC| = 2a$. Assume $|AC| = b < \pi/2$. Let K and M be the midpoints of AB and BC respectively, denote $t = |KM|$.

(a) Find $\cos t$ in terms of a and b.

(b) Find $\cos b$ assuming $t = a$. Which of the following is true in this case: $b < 2t$, $b = 2t$ or $b > 2t$? Justify your answer.

Q4 Let $ABCD$ be a trapezoid on the Euclidean plane, with side BC parallel to AD. Suppose that $|AD| = 2|BC|$. Let M and N be the midpoints of AD and BC respectively. Let P be the point of intersection of the diagonals of $ABCD$.

(a) Show that the points M, N, P are collinear.

(b) Find the ratio $|MP|/|PN|$.

SECTION B

Q5 The circles C_1 and C_3 on the Euclidean plane are tangent at a point A from the outside. The circles C_2 and C_4 are also tangent to each other at the point A from the outside, and each cross C_1 and C_3 orthogonally at A. The points P_i, $i = 1, 2, 3, 4$, are the intersection points of C_i with C_{i+1} distinct from A (here, we assume $C_5 = C_1$).

(a) Show that there exists a circle C passing through the points P_1, P_2, P_3, P_4.

(b) Is it true that the group of Möbius transformations acts transitively on all the configurations of four circles described as above? Justify your answer.

(c) Find $[P_1, P_2, P_3, P_4]$ assuming that the circle C described in part (a) is bisecting the angle between C_1 and C_2.

Q6 Recall that a polygon P on the hyperbolic plane is called ideal if all vertices of P lie at the boundary $\partial\mathbb{H}^2$.

(a) Let ABC be an ideal triangle on the hyperbolic plane. Let γ_0 be the inscribed circle for $\triangle ABC$. Let γ_1 be a circle tangent to γ_0 and to exactly two sides of $\triangle ABC$. Find the radius of γ_1.

(b) Let $ABCD$ be an ideal quadrilateral on $\mathbb{H}^2$, let l and m be the common perpendiculars to the pairs of opposite sides of $ABCD$. Is it always true that the intersection point of l and m coincides with the intersection points of the diagonals? Justify your answer.

(c) Let P be an ideal pentagon, let $G \subset Isom(\mathbb{H}^2)$ be the group of symmetries of P. Let F be a fundamental domain of the action of G on P. Assuming that $|G| = 10$, i.e. that P is a regular ideal pentagon, find the area of F.

Q7 (a) Find the cross-ratio of the following four points in $\mathbb{R}P^2$ (given in homogeneous coordinates):

$$A = (1:1:0), \quad B = (1:1:1), \quad C = (1:1:2), \quad D = (0:0:1).$$

(b) Let $A_1A_2A_3$ be a triangle on $\mathbb{R}P^2$, let P and Q be two points such that $A_i \notin PQ$ for $i = 1, 2, 3$. Denote $B_{i,j} = PA_i \cap QA_j$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. Show that if the points B_{12}, B_{23}, B_{31} are collinear, then the lines $A_1B_{32}, A_2B_{13}, A_3B_{21}$ are concurrent.

(c) Formulate the statement dual to the one given in part (b).

Q8 Let $s : S^2 \to \mathbb{R}^2$ be the stereographic projection taking the sphere to a plane through its center. Let i be an orientation-preserving isometry of the sphere S^2.

(a) Show that $\psi = s \circ i \circ s^{-1} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a Möbius transformation.

(b) Is it true or false that every Möbius transformation can be obtained as $s \circ i \circ s^{-1}$ for a suitable isometry $i \in Isom^+(S^2)$? Justify your answer.

(c) Let i_0 be a rotation of the sphere by $\pi/2$ around some axis. Find the type of the Möbius transformation $\psi_0 = s \circ i_0 \circ s^{-1}$. Justify your answer.