Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3251-WE01

Title:
Stochastic Processes III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Let $X_1 \sim \text{Bernoulli}(p)$ and $X_2 \sim \text{Poisson}(\lambda)$ where $p \in (0,1)$ and $\lambda > 0$.

(a) Show that $X_1 \leq_{\text{st}} X_2$ if and only if $\lambda \geq -\log(1-p)$.

(b) Suppose $\lambda \geq -\log(1-p)$. Show that $d_{\text{TV}}(X_1, X_2) = 1 - e^{-\lambda} - \min(p, \lambda e^{-\lambda})$.

Q2 This question deals with Poisson processes.

(a) Customers arrive at a store according to a Poisson process of rate 5/hour. Each customer is independently a little spender with probability 2/3 or a big spender with probability 1/3. A little spender spends on average 3 pounds and a big spender spends on average 9 pounds. Let T be the total amount of money earned by the shop in the first 10 hours. Find $\mathsf{E}[T]$.

(b) Consider two independent Poisson processes consisting of red balls and blue balls, both having rate λ. Find the probability that 4 red balls appear before 3 blue balls appear.

Q3 This question deals with Martingales.

(a) State the definition of a Martingale sequence. Make sure to state all probabilistic objects and conditions involved in the definition.

(b) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common distribution $\mathsf{P}(X_k = +1) = p$, $\mathsf{P}(X_k = -1) = 1 - p = q$ where $0 < p < 1$. Define

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} X_k \quad n \geq 1.$$

Let $\mathcal{F}_0$ be the trivial σ-algebra and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \geq 1$. Find a constant c such that the process $M_n = S_n + cn$ for $n \geq 0$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$. Make sure to verify all the martingale conditions.

Q4 Let $(Z_n)_{n \geq 0}$ be a branching process with $Z_0 = 1$ and suppose $\varphi_n(s) := \mathsf{E}[s^{Z_n}]$ satisfies $\varphi_1(s) := \mathsf{E}[s^{Z_1}] = \frac{1}{3}(1 + s + s^2)$ for any $s > 0$.

(a) Find $\mathsf{E}[Z_n]$ and $\mathsf{E}[Z_n^2]$ in terms of n.

(b) Find $\text{Cov}(Z_{2023}, Z_{2024})$.

SECTION B

Q5 Let $(S_n)_{n\geq 0}$ be a random walk starting from 0 with i.i.d. increments $S_{n+1} - S_n \overset{(d)}{=} X$ satisfying

$$\mathsf{P}(X = j) = \begin{cases} p(1-p)^{j-1} & \text{for integers } j \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

for some $p \in (0,1)$. Consider

$$C(t) := \sum_{n\geq 1} 1_{\{S_n \leq t\}} \quad \text{for } t \geq 0 \qquad \text{and} \qquad \widehat{C}(u) := \sum_{n\geq 1} u^{S_n} \quad \text{for } u \in (0,1).$$

(a) Find a formula for $\widehat{c}(u) := \mathsf{E}[\widehat{C}(u)]$ in terms of u and p.

(b) Using (a), prove that $c(t) := \mathsf{E}[C(t)] < \infty$ for any fixed $t > 0$.
(Hint: for any $x \geq 0$ and $t > 0$, we have $1_{\{x\leq t\}} \leq e^{1-x/t}$.)

(c) Explain why $C(t) \leq t$ for all $t \geq 0$, then show that

$$\lim_{t\to\infty} \frac{C(t)}{t} = p \quad \text{a.s.} \qquad \text{and} \qquad \lim_{t\to\infty} \frac{c(t)}{t} = p.$$

(d) Using (a) or otherwise, find a formula for $c(j)$ for any integers $j \geq 0$.
(Hint: find a probabilistic representation for the coefficients $(\widehat{c}_n)_n$ in the Taylor series expansion $\widehat{c}(u) = \sum_{n\geq 0} \widehat{c}_n u^n$.)

Q6 (a) Let $(U_n, V_n)_{n\geq 1}$ be independent pairs of non-negative random variables with $U_n \leq_{\mathrm{st}} V_n$ for all n. Suppose A and B are two non-negative integer-valued random variables on the same probability space that are independent of $(U_n, V_n)_{n\geq 1}$ and such that $A \leq_{\mathrm{st}} B$. Prove that

$$\sum_{n\leq A} U_n \leq_{\mathrm{st}} \sum_{n\leq B} V_n.$$

(b) Let $p \in (0,1)$. Suppose $M \sim \mathrm{Binomial}(10, p)$, $L \sim \mathrm{Binomial}(5, p^2)$ and $R \sim \mathrm{Binomial}(5, 1-(1-p)^2)$. Show that $2L \leq_{\mathrm{st}} M$ and $M \leq_{\mathrm{st}} 2R$.
(Hint: you may want to construct a suitable coupling using i.i.d. Bernoulli(p) random variables $C_1, C_2, \ldots, C_{10}$.)

(c) Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two branching processes with $X_0 = Y_0 = 1$, and suppose their offspring distributions are described by the generating functions

$$\varphi^X(s) := \mathsf{E}[s^{X_1}] = \left(\frac{1+3s^2}{4}\right)^5 \quad \text{and} \quad \varphi^Y(s) := \mathsf{E}[s^{Y_1}] = \left(\frac{1+s}{2}\right)^{10}.$$

Which of the two processes is more likely to survive forever? Using the previous parts, justify your claim with a complete proof.

Q7 A barbershop has one chair for a barber to cut customers' hair and two chairs in the waiting room. The barber cuts hair at a rate of 3 (people/hour). Customers arrive at a rate of 2 (people/hour). Customers leave if both chairs in the waiting room are occupied. Let $X(t)$ denote the number of customers in the barbershop at time t (this includes the barber's chair and the waiting room). The process $X(t)$ is a continuous time Markov process.

7.1 Find the state space of $X(t)$ and its generator (Q-matrix). Explain your reasoning in probabilistic language.

7.2 Show that $X(t)$ is an irreducible Markov process.

7.3 Find the stationary distribution of $X(t)$. Explain what proportion of customers are lost from service in the long run.

7.4 Find $\lim_{t\to\infty} p_{0,1}(t)$ with appropriate justification.

Q8 Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution

$$\mathsf{P}(X_i = 0) = \mathsf{P}(X_i = 1) = 1/2$$

Consider the (random) infinite sequence $X_1, X_2, X_3, \ldots$. Let T be the first time the pattern 1010 appears in the sequence. For instance, if the sequence starts off as $11001010\ldots$ then the pattern appears at time $T = 8$. Find $\mathsf{E}[T]$. Justify all steps in the calculation and quote the theorems that you use. (Hint: use martingales)