EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3301-WE01

Title:
Mathematical Finance III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Consider the market $\mathcal{M} = (B_t, S_t)$ in which the risk-free asset B_t has price dynamics $B_t = 1.05^t$, $t = 0, 1, 2$, and the risky asset S_t has current price $S_0 = 100$ and evolves according to the binomial model with $u = 1.1$ and $d = 0.95$.

(a) Calculate the risk-neutral measure in this market.

(b) Find the no-arbitrage prices at times $t = 0, 1$ of a lookback option whose payoff is given by

$$X = (S_{max} - 100)^+$$

at time $T = 2$. Here $S_{max}(\omega) = \max(S_0(\omega), S_1(\omega), S_2(\omega))$.

Q2 (a) State the put-call parity formula relating the fair prices at time 0 of a European call option with expiry date T and strike price K, and a European put option with the same parameters.

(b) Consider the two-period binomial market in which $S_0 = 64$, $r = 0.2$, $u = 1.25$ and $d = 0.75$. Calculate the fair price at time 0 for a European call option with strike price 54 and expiry date $T = 2$, and use put-call parity to determine the fair price at time 0 for a European put option with strike price 54 and expiry $T = 2$.

(c) Does the put-call parity formula apply to American options? Justify your answer with a suitable explanation.

Q3 (a) State the definition of *Brownian motion*;

(b) Let $(W_t)_{t\geq 0}$ be a Brownian motion and $a > 0$. Prove that

$$\left(\frac{-1}{\sqrt{a}} W_{at}\right)_{t\geq 0} \text{ is a Brownian motion;}$$

(c) State the definition for a stochastic process $(X_t)_{t\geq 0}$ to be *a martingale with respect to a given filtration* $(\mathcal{F}_t)_{t\geq 0}$;

(d) Let $(W_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by $(W_t)_{t\geq 0}$. Prove that

$$(W_t^2 - t)_{t\geq 0} \text{ is a martingale with respect to } (\mathcal{F}_t)_{t\geq 0}.$$

Q4 Let $(W_t)_{t\geq 0}$ be a Brownian motion.

(a) State the rules of Itô calculus (box calculus) in terms of $\mathrm{d}t$ and $\mathrm{d}W_t$, and state also the Itô formula for $\mathrm{d}F(t, S_t)$ for a smooth function $F(t, x)$ and Itô process $(S_t)_{t\geq 0}$;

(b) Apply the Itô formula to $\mathrm{d}(W_t^4)$ and express $\mathrm{d}(W_t^4)$ in terms of $\mathrm{d}t$, $\mathrm{d}W_t$ and W_t;

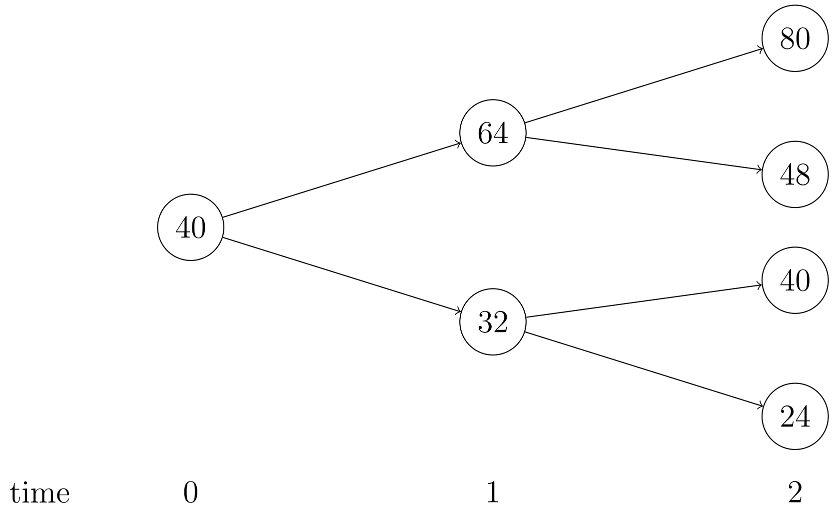
(c) Prove that

$$\mathbb{E}[W_t^6] = 15t^3.$$

SECTION B

Q5 (a) A *Bermudan option* is an option which can only be exercised at certain predetermined times $T_1, T_2, \ldots, T_n = T$.

Consider the two-period binomial market, in which the risk-free interest rate is $r = \frac{1}{5}$ and the risky asset has the following price dynamics.



In the context of this market, calculate the fair prices at time 0 for the following options.

(i) a European put option with strike price $K = 44$ and expiry time $T = 2$

(ii) an American put option with strike price $K = 44$ and expiry time $T = 2$

(iii) a Bermudan put option, which can be exercised at times $t = 0$ and $t = 2$, with strike price $K = 44$.

(b) Order your answers from the previous part from smallest to largest. In general, which of the three types of put option (European, American, and Bermudan) has the smallest price, and which has the largest? Justify your answer.

If instead, we had calculated the prices of European, American, and Bermudan *call* options, would you find the same inequalities? Justify your answer.

Q6 In this question, all call and put options are European, and have expiry date T.

(a) For each of the following portfolios, draw the graph of the payoff $\Phi(S_T)$ against S_T.

(i) Two put options with strike price K, plus one share.

(ii) One call option with strike price K, plus one call option with strike price L, where $K < L$.

(iii) One put option with strike price K, short one call option with strike price L. You should consider the cases $K < L$ and $L < K$.

(b) Consider the following three portfolios.

- Portfolio X consists of two call options with strike price $L = \frac{1}{2}(K_1 + K_2)$.
- Portfolio Y consists of one call option with strike price K_1 and one call option with strike price K_2.
- Portfolio Z consists of one call option with strike price $K_1 - 5$ and one call option with strike price $K_2 + 5$.

Prove that the prices of these portfolios, X_t, Y_t, and Z_t, are in the relation $X_t \leq Y_t \leq Z_t$ for any $t \leq T$. Carefully justify your answer, and state any results you use from the course.

Q7 Let $B_t = e^{rt}$ be the bond price with interest rate $r > 0$ and $(S_t)_{t\geq 0}$ be the stock price following the Black-Scholes-Merton SDE:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad S_0 = x_0 > 0, \quad t \in [0,T] \ .$$

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by $(W_t)_{t\in[0,T]}$.

(a) Let (a_t, b_t) be a portfolio and $V_t = a_t B_t + b_t S_t$ be the value process. State the definitions for (a_t, b_t) being *self-financing* and for (a_t, b_t) *replicating a contingent claim* Φ *at expiry time* T;

(b) Verify that $S_t = S_0 \mathrm{e}^{(\mu-\sigma^2/2)t+\sigma W_t}$ is a solution to $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$;

(c) For a smooth function $F : [0,T] \times \mathbb{R} \to \mathbb{R}$, prove

$$\mathrm{d}F(t,S_t) = \left[\partial_t F(t,S_t) + \mu S_t \partial_x F(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx} F(t,S_t)\right] \mathrm{d}t + \sigma S_t \partial_x F(t,S_t) \mathrm{d}W_t \ ;$$

(d) Let $\Pi_t(\Phi)$ be the fair price (no-arbitrage price) of the contingent claim Φ at time t, and let $F(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a smooth function so that $F(t,S_t) = \Pi_t(\Phi)$. Prove that the hedging portfolio (a_t, b_t) can be expressed by

$$a_t = \frac{F(t,S_t) - S_t \partial_x F(t,S_t)}{B_t} \ , \quad b_t = \partial_x F(t,S_t) \ ,$$

and also prove that $F(t,x)$ is the solution to the following partial differential equation:

$$\begin{cases} \partial_t F + \frac{1}{2}\sigma^2 x^2 \partial_{xx} F + rx\partial_x F - rF = 0 \\ F(T,x) = \Phi(x) \ . \end{cases}$$

Q8 Let $B_t = e^{rt}$ be the bond price with interest rate $r > 0$ and $(S_t)_{t\geq 0}$ be the stock price following the Black-Scholes-Merton SDE:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad S_0 = x_0 > 0, \quad t \in [0, T] \ .$$

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by $(W_t)_{t\in[0,T]}$. Let $(\widetilde{W}_t)_{t\geq 0}$ be the Brownian motion under the risk-neutral measure (martingale measure) $\mathbb{Q}$.

(a) Write down the formula for $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ in terms of T, W_T, μ, σ, r;

(b) Let Φ be a contingent claim with expiry time T. Let $X_t = \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-rT}\Phi \mid \mathcal{F}_t]$. Prove that there exists a process $(H_t, t \geq 0)$ satisfying $\mathrm{d}X_t = H_t \mathrm{d}\widetilde{W}_t$;

(c) Let $\Phi(S_T)$ be a contingent claim with $\Phi(x) = x - K$ $(K > 0)$ and expiry time T. Compute the fair price $\Pi_t(\Phi)$ at time t by using the risk-neutral valuation formula and express it in terms of K, r, t, S_t and T;

(d) Under the same setting as (c), compute the hedging portfolio (a_t, b_t).