EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3391-WE01

Title:
Quantum Computing III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Consider the Hilbert space of a single qubit. Use the standard basis in which

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

1.1 Explain whether each of the following operators correspond to an observable:

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Can any of these be measured simultaneously? Explain your answer.

1.2 For each of the operators above which correspond to observables, compute the possible outcomes of a measurement when the state of the system is given by

$$|\psi\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

1.3 A measurement is made corresponding to the operator

$$U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

with the system in the state $|\psi\rangle$ above. The outcome of the measurement is 0. If we now measure R, what is/are the possible outcome(s)?

Q2 You are given the matrix

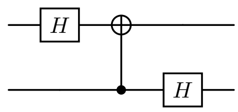
$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} - \frac{i}{8} \\ \frac{1}{8} + \frac{i}{8} & \frac{1}{2} \end{pmatrix}.$$

2.1 Explain why this can be a density matrix for a qubit system.

2.2 Is the system in a pure or a mixed state?

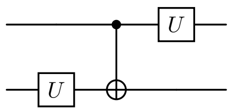
2.3 Compute the Von Neumann entropy for this system.

Q3 Consider the circuit



3.1 Give the action of this circuit on the computational basis states as a 4×4 unitary matrix.

3.2 Find single-qubit unitary U such that the following circuit is equivalent to the one you just studied (note that the control and target bits in the CNOT gate are swapped):



Q4 **4.1** Is it possible to construct a 2-qubit code that corrects single-qubit flip errors? If so, give the code and error correction procedure explicitly. If not, prove that it is impossible.

4.2 It is possible to protect against single-qubit bit flip errors by using six qubits, using the following encoding:

$$\left|\overline{0}\right\rangle := |000000\rangle \qquad ; \qquad \left|\overline{1}\right\rangle := |111111\rangle \ .$$

Construct logical $\overline{X}$ and $\overline{Z}$ gates, and determine whether the logical gates you have constructed are fault-tolerant.

SECTION B

Q5 Consider a 2-qubit system with qubits A and B, described by a density matrix given by

$$\rho = \frac{1}{2}\Big(|00\rangle \langle 00| + a\,|01\rangle \langle 01| + b\,|11\rangle \langle 11| \Big)\,.$$

5.1 What are the conditions on a and b for this to be a density matrix?

5.2 Compute the reduced density matrices ρ_A and ρ_B.

5.3 Compute the Von Neumann entropy $S(A,B)$ for ρ and the entanglement entropies $S(A)$ and $S(B)$.

5.4 Compute the quantity $S(A,B) - S(A)$ for the special values $a = 1, b = 0$. Explain what your result means in terms of the information obtained by measuring qubit A.

Q6 Alice and Bob share a Bell state, and Alice also has an unknown state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$, so that the full state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,|\psi\rangle \otimes \Big(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \Big)\,.$$

in which Alice controls $|\psi\rangle$ and the first qubit of the remaining two.

6.1 Define the reduced density matrix of the first qubit as usual,

$$\rho_1 = \text{Tr}_{23}(\rho)\,. \tag{1}$$

What is the value of the trace of the square of this reduced density matrix, so

$$\text{Tr}_1\Big(\rho_1^2\Big)\,?$$

6.2 Alice wants to entangle her unknown state $|\psi\rangle$ with the state of her second qubit. She applies the unitary operator that takes

$$|00\rangle \rightarrow |00\rangle\ , \quad |01\rangle \rightarrow |01\rangle\ , \quad |10\rangle \rightarrow |11\rangle\ , \quad |11\rangle \rightarrow |10\rangle\ ,$$

to her two qubits. What is the state of the system now?

6.3 Compute the reduced density matrix ρ_1 defined in (1) to argue that the above operation indeed managed to entangle her two qubits.

6.4 She now applies the unitary transformation

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle \Big)\,, \quad |1\rangle \rightarrow \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle \Big)\,,$$

on her *first* qubit only, and then measures her two qubits using the observable $|1\rangle \langle 1|$ for both. She finds the value 0 twice. Is Bob's qubit state now mixed or pure? Give the state of his qubit.

Q7 We define the following unitary transformation acting on the two-qubit Hilbert space in the computational basis:

$$U := \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

7.1 Write U as a product of unitaries U_{ij}, where each U_{ij} acts non-trivially only on the two dimensional subspace spanned by the computational basis elements $|i\rangle$ and $|j\rangle$.

7.2 Decompose the U_{ij} matrices you found in the previous part into controlled-unitary gates and unitary transformations acting on single qubits.
[***Hint**: if you need to use a Gray code, use* $01 \to 00 \to 10$.]

7.3 Simplify, explaining the intermediate steps, the resulting circuit as much as you can. You do not need to decompose controlled-unitaries into single qubit unitaries and CNOTs.
[***Hint**: you can simplify the circuit into a NOT gate and a controlled-unitary.*]

Q8 **8.1** Show that

$$H^n |s\rangle = \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} (-1)^{s \cdot k} |k\rangle$$

with $s \cdot k := s_{n-1}k_{n-1} + \ldots + s_0 k_0 \mod 2$ the bitwise product.

8.2 Construct, using only gates in the universal gate set $\{H, T, \text{CNOT}\}$ (not necessarily all of them), a circuit acting on n qubits that transforms the input state with all bits initialised to 0, namely $|0\rangle \otimes \cdots \otimes |0\rangle$, into the alternating superposition of computational basis states

$$\frac{1}{2^{n/2}} \sum_{k=0}^{2^k-1} (-1)^k |k\rangle \ .$$

[***Hint**: It might be useful to recall that* $T^4 = Z := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.]