EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3401-WE01

Title:

Cryptography and Codes III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

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SECTION A

Q1 In this question, messages are sequences of digits regarded as integers modulo 10.

(a) A 2×2 Hill cipher modulo 10 is used to encrypt the plaintext '1989'. The resulting ciphertext is '9532'. Find the encryption and decryption keys.

(b) A 3×3 Hill cipher modulo 10 is used to encrypt the plaintext '124'. The resulting ciphertext is '783'. Find the encryption of the plaintext '248'.

Q2 Let E be the elliptic curve $y^2 = x^3 + 2x$ over $\mathbb{F}_7$.

(a) Find all 2-torsion points of $E(\mathbb{F}_7)$.

(b) Show that $(5, 4) \in E(\mathbb{F}_7)$ has order 8.

(c) Is $E(\mathbb{F}_7)$ cyclic?

Q3 (a) How many ternary cyclic codes of block-length 4 are there? Justify your answer.

(b) For each such code give a generator-matrix, and the generator-polynomial.

Q4 (a) Let C be a linear code in $\mathbb{F}_q^n$. Give the definition of its dual $C^\perp$. State, without proof, how the dimension of C is related to that of $C^\perp$. Do the same for their generator and check-matrices.

(b) Let now D be another linear code in $\mathbb{F}_q^n$ and define

$$C + D := \{c + d \in \mathbb{F}_q^n \mid c \in C,\ d \in D\}.$$

Show that $(C + D)^\perp = C^\perp \cap D^\perp$. (Here you may assume without proof that $C + D$ is a linear code in $\mathbb{F}_q^n$.)

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SECTION B

Q5 Alice and Bob use the Diffie–Hellman key exchange protocol. They publicly agree on a large[1] prime p and primitive root $g \in \mathbb{F}_p^\times$. Alice chooses an integer $0 \leq \alpha < p-1$ and Bob chooses an integer $0 \leq \beta < p-1$. They keep these secret and exchange the values of g^α and g^β over a public channel.

(a) What is their shared secret key? How can they each calculate it?

Now Alice and Bob generate secret sequences of integers $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ by taking

$$\begin{aligned} \alpha_1 &= \alpha \\ \beta_1 &= \beta \\ \alpha_{n+1} &= a\alpha_n + b \bmod (p-1) \\ \beta_{n+1} &= c\beta_n + d \bmod (p-1) \end{aligned}$$

where a, b, c, d are publicly known integers with a and c coprime to $p-1$. They use these to obtain shared secrets $\kappa_1, \kappa_2, \ldots$ using the Diffie–Hellman protocol.

(b) Suppose that Eve observes the messages sent between Alice and Bob and also obtains the value of κ_2. Show that she can find κ_n for all $n \geq 2$.

(c) Can Eve find κ_1 (in a reasonable amount of time)? Justify your answer.

(d) Can Eve find α and β (in a reasonable amount of time)? Justify your answer.

Q6 Alice's RSA public key is (n, e) where n is a product of two primes.

(a) Suppose that $(n, e) = (399797, 3)$. Bob sends the message m to Alice. Its encryption is 8000. Find m.

(b) Suppose that $(n, e) = (18871, 17)$. Use Fermat's method to factorise n and hence find the decryption exponent d.

(c) Suppose that $n = 12449$. Let $P = (2, 5)$ be a point on the elliptic curve $E : y^2 = x^3 + 17$ modulo n. By attempting to compute $[4]P$, factorise n.

[1] Say, $p > 2^{2048}$.

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Q7 Let $g(x)$ be the generator-polynomial of a binary cyclic code C of length $n > 1$ and dimension at least one.

(a) Show that, if $g(x)$ has $x-1$ as a factor then the code C contains no codewords of odd weight.

(b) Show that if $x-1$ is not a factor of $g(x)$ then the code C contains the codeword consisting of all 1s.

(c) Assume that for any $m \in \mathbb{N}$ with $m < n$, $g(x)$ does not divide $x^m - 1$. Show that the code C has minimum distance at least 3.

(d) Suppose $g(x)$ is such that the corresponding code C contains both even-weight and odd-weight codewords. Show that the polynomial $(x-1)g(x)$ also generates a binary cyclic code C_1 of length n, and that $C_1 = \{c \in C \mid w(c) \text{ is even}\}$. That is, the code C_1 consists of the even-weight codewords of C.

Q8 (a) Let $q \geq n \geq k \geq 0$ be positive integers and $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{F}_q^n$, with the a_j all distinct and the b_j all non-zero. Give the definition of the Reed-Solomon Code $\mathrm{RS}_k(\mathbf{a}, \mathbf{b})$ as a q-ary $[n, k]$ code. Furthermore give, without proof, the minimum distance of this code and the form of a generator-matrix.

(b) Consider the polynomial $x^2 + 1 \in \mathbb{F}_3[x]$. Show that it is irreducible. Is it primitive? Justify your answer.

(c) Consider the field $\mathbb{F}_9 = \mathbb{F}_3[x]/(x^2 + 1)$. Give a cyclic 9-ary $[4, 2, 3]$ code by giving both a generator-matrix and its generator-polynomial.

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