Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3421-WE01

Title:
Bayesian Computation and Modelling III

Time:	2 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Consider a trivariate target density $\pi(\underline{\theta}) = \pi(\theta_1, \theta_2, \theta_3)$ and a deterministic scan Gibbs sampling algorithm for sampling from this target.

1.1 Briefly describe the main steps of the deterministic scan Gibbs sampler at iteration j for this target.

1.2 Write down the form of the transition density $p(\underline{\phi}|\underline{\theta})$ of the resulting Markov chain, giving your answer in terms of full conditional densities of the form $\pi(\cdot|\cdot,\cdot)$.

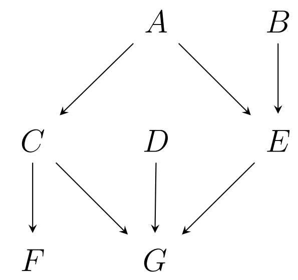
1.3 Briefly explain the importance of updating all components.

1.4 Suppose that

$$\pi(\theta_1, \theta_2, \theta_3) \propto \theta_3^{3/2} \exp\left\{ -\frac{1}{2}\big[\theta_1^2 + \theta_2^2 - 2\theta_2 + \theta_3(8 - 4\theta_1 - 4\theta_2 + 2\theta_1\theta_2 + \theta_1^2 + \theta_2^2)\big] \right\},$$
$$\theta_1, \theta_2 \in \mathbb{R},\ \theta_3 > 0.$$

Derive and identify all full conditional distributions for this trivariate target.

Q2 Consider the following directed acyclic graph that encodes the conditional independence relationships between seven variables.



2.1 Which of the variables are exogenous and which are endogenous?

2.2 Write down two sets of variables that would each D-separate B and G on their own.

2.3 A researcher has prior beliefs about A. They believe that the mean value is around 3 and that the standard deviation is around 1. If they further believe that the distribution is right-skewed and bounded between 0 and 10, give a suitable prior distribution for A.

2.4 The researcher finds out that D, E, F and G are going to be studied in each of three independent regions and that, within each region, there will be ten independent observations of both F and G. Redraw the directed acyclic graph to accommodate this information.

SECTION B

Q3 Consider a hidden Markov model (HMM) for data $y_0, \ldots, y_T$ generated via a hidden Markov chain $X_0, \ldots, X_T$ with a transition density of $p(x_t|x_{t-1})$ $(t = 1, \ldots, T)$ and initial density $p(x_0)$. Denote the data likelihood by $p(y_t|x_t)$ $(t = 0, \ldots, T)$.

3.1 Derive an expression for $p(x_t, y_t|y_{0:t-1})$ given $p(x_{t-1}|y_{0:t-1})$. Be sure to give the full expressions for terms before they are simplified through the special structure of the hidden Markov model.

3.2 Use Bayes Theorem to derive the full, normalised expression for $p(x_t|y_{0:t})$ from $p(x_t, y_t|y_{0:t-1})$. **Hint:** first write $p(x_t|y_{0:t})$ as $p(x_t|y_{0:t-1}, y_t)$.

3.3 A bootstrap particle filter is applied to this HMM resulting in a weighted sample at time t, $\left\{(x_t^{(k)}, \tilde{w}_t^{(k)})\right\}_{k=1}^N$.

(a) Write down the density that the weighted samples are an approximation to.

(b) The samples $\left\{x_t^{(k)}\right\}_{k=1}^N$ are resampled (with replacement) with probabilities equal to their weights. Denote the resulting unweighted samples by $\left\{\tilde{x}_t^{(k)}\right\}_{k=1}^N$. Write down the density that these unweighted samples are an approximation to.

(c) For each sample $\tilde{x}_t^{(k)}$ from (b), a new sample $x_{t+1}^{(k)}$ is generated from the transition density $p(x_{t+1}|\tilde{x}_t^{(k)})$. Write down the density that these new unweighted samples are an approximation to.

(d) Using the samples from (c), the following quantity is constructed:

$$\frac{\sum_{k=1}^N p\left(y_{t+1} \,\middle|\, x_{t+1}^{(k)}\right) x_{t+1}^{(k)}}{\sum_{k=1}^N p\left(y_{t+1} \,\middle|\, x_{t+1}^{(k)}\right)}.$$

What is this quantity estimating?

Q4 We have the following model of a relationship between observed x_i and y_i:

$$\begin{aligned} \widetilde{y}_i &= \alpha + \beta \widetilde{x}_i, \\ y_i &= \widetilde{y}_i + \epsilon_{yi}, \\ x_i &= \widetilde{x}_i + \epsilon_{xi}, \\ \epsilon_{yi} | \sigma_y^2 &\overset{\text{iid}}{\sim} \mathrm{N}(0, \sigma_y^2), \\ \epsilon_{xi} | \sigma_x^2 &\overset{\text{iid}}{\sim} \mathrm{N}(0, \sigma_x^2), \end{aligned}$$

for $i = 1, \ldots, n$ and with $\epsilon_{xi} \perp \epsilon_{yj}$ for all i and j.

4.1 Write down the joint likelihood function for the model parameters, α, β, σ_x^2 and σ_y^2, with respect to n observed (x_i, y_i)-pairs.

4.2 Which of the parameters are not identifiable? Give one suggestion of how this issue can be resolved.

4.3 We will only receive a value for the y_i if it is greater than three. Write down the likelihood when we have n observed and complete (x_i, y_i)-pairs and m observed (x_i, y_i)-pairs with the y_i value missing due to their value being less than three.

4.4 In performing a Bayesian analysis of this model, we will utilise a folded distribution for the prior for α. Let $f(\alpha)$ be a probability density function that is positive for all $\alpha \in \mathbb{R}$. A folded version of $f(\alpha)$ at boundary a, $g(\alpha)$ say, is

$$g(\alpha) = \begin{cases} f(\alpha) + f(2a - \alpha) & \alpha > a, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $g(\alpha)$ is a proper density function.

(b) Suggest a suitable proposal distribution for α if a Metropolis-Hastings sampler was to be employed when utilising a folded prior. Give your reasoning.

4.5 For this model, an expert believes *a priori* that there is a dependency between α and β. Outline a strategy for eliciting beliefs about the two parameters from a single expert.