Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH3471-WE01

Title: Geometry of Mathematical Physics III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 $SO(3)$ contains the real 3×3 matrices O for which $O^T = O^{-1}$ and $\det O = 1$.

1.1 Show that $SO(3)$ is a group.

1.2 Show that acting on a vector $\mathbf{x} \in \mathbb{R}^3$ as $\mathbf{x} \to O\mathbf{x}$ leaves the inner form

$$|\mathbf{x}|^2 := \mathbf{x} \cdot \mathbf{x}$$

invariant.

1.3 Find the Lie algebra element ℓ associated with the path

$$t \mapsto \begin{pmatrix} \cos at & \sin at & 0 \\ -\sin at & \cos at & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3) \ .$$

Here $a \in \mathbb{R}$ and $t \in [-1, 1]$.

1.4 Compute $e^{\ell\phi}$ for $\phi \in \mathbb{R}$ and ℓ defined in the previous sub-question.

Hint:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \ ,$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \ .$$

Q2 Decide if each of the following defines a representation of the respective groups, and explain your reasoning.

2.1 $r_1 : S \mapsto S^2$ for $S \in U(1)$.

2.2 $r_2 : S \mapsto S^2$ for $S \in SO(n)$.

2.3 $r_3 : S \mapsto \sqrt{S}$ for $S \in U(1)$.

2.4 $r_4 : g \mapsto \det g$ for $g \in U(n)$.

2.5 $r_5 : g \mapsto (\det g)^{-1}$ for $g \in U(n)$.

Q3 Let L_3 be the Lorentz group in three dimensions of space-time, i.e. L_3 is the set of linear maps $\{\Lambda\}$ acting on (x_0, x_1, x_2) with the property that

$$|\mathbf{x}| := -(x^0)^2 + (x^1)^2 + (x^2)^2 \ ,$$

is invariant under

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \ .$$

3.1 Find the conditions that $\Lambda^\mu{}_\nu$ needs to obey to be in L_3.

3.2 Work out the Lie algebra $\mathfrak{l}_3$ of L_3. [Hint: $\mathfrak{l}_3$ is three-dimensional]

3.3 Is the exponential map for L_3 surjective? Explain your reasoning.

Q4 The field strength tensor of electromagnetism is given in terms of electric and magnectic fields by

$$F^{\mu\nu} := \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} .$$

Write the following expressions in terms of electric and magnetic fields:

4.1 $F_{\mu\nu}$.

4.2 $F^{\mu\nu}F_{\mu\nu}$.

4.3 $F^{\mu\nu}F^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma}$.

SECTION B

Q5 Consider the adjoint representation of $SU(2)$, in which $SU(2)$ acts on $\mathfrak{su}(2)$.

5.1 Write down the adjoint action of $g \in SU(2)$ on $\gamma \in \mathfrak{su}(2)$.

5.2 Let

$$g(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} .$$

Show that $G := \{g(\phi)|\phi \in [0, 2\pi)\}$ is a $U(1)$ subgroup of $SU(2)$.

5.3 Explain why the adjoint representation of $SU(2)$ also defines a representation r of G, and describe the action of this representation on $\mathfrak{su}(2)$.

5.4 Decompose the representation r defined in the previous sub-question into irreducible representations r_i.

5.5 Find the associated Lie algebra representation of each of the irreducible representations r_i found in the previous sub-question.

Q6 We can act with $g \in SU(n)$ on the vector space V of complex $n \times n$ matrices by letting

$$F(g) : M \mapsto g^\dagger M g \,,$$

for $M \in V$.

6.1 Show that this defines a representation r of $SU(n)$.

6.2 Show that the set W of matrices M proportional to the identity matrix form an invariant subspace under r.

6.3 Check that r is a unitary representation with respect to the inner form

$$|M|^2 := \sum_{i,j} \overline{M}_{ij} M_{ij} \,.$$

6.4 Use the results above to find another invariant subspace $W' \in V$.

6.5 For $n = 2$, show that restricting r to W' results in an irreducible representation.

Q7 Consider a $U(1)$ gauge theory in 3 dimensions given by the action

$$S = \int d^3x \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

7.1 How many independent components does $F_{\mu\nu}$ have ?

7.2 Describe the gauge symmetry of the system and find the equations of motion.

7.3 It is possible to choose a gauge where $\partial_\mu A^\mu = 0$. Writing

$$A_\mu = p_\mu e^{ik_\nu x^\nu}$$

find the relations that need to be obeyed by p_μ and k_μ to satisfy the gauge condition and solve the equations of motion.

7.4 Find the residual gauge symmetry left after imposing $\partial_\mu A^\mu = 0$. In other words, are there gauge transformations which respect the condition $\partial_\mu A^\mu = 0$, and if so what are they?

7.5 Using the residual gauge symmetry, how many physical choices of p_μ remain ?

7.6 Let $\partial_\mu \phi = \epsilon_{\mu\nu\rho} F^{\nu\rho}$ for a real scalar field ϕ. Find the action for ϕ.

Q8 Consider a field theory with three real scalar fields ϕ_i, $i = 1, 2, 3$ and action

$$S_1 = \int d^4x \; \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i \; + \; \frac{1}{2} m^2 \, \phi_i \phi_i \,,$$

where we are using summation convention both for the Lorentz index μ and the index i, and m^2 is a constant.

8.1 Find the equations of motion following from S_1.

8.2 Show that S_1 is invariant under Lorentz transformations.

8.3 Show that letting $\phi_i \to O_{ij} \phi_j$, with O_{ij} the components of a matrix $O \in O(3)$, leaves S_1 invariant.

Now we introduce two more complex scalar fields χ_k, $k = 1, 2$, and let the action of our system be $S = S_1 + S_2$ with

$$S_2 = \int d^4x \; \partial_\mu \bar{\chi}_k \partial^\mu \chi_k + \lambda \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}^\dagger \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} .$$

8.4 Find the equations of motion following from S.

8.5 For $\lambda = 0$, S has a $SU(2)$ symmetry under which (χ_1, χ_2) transform in the **2** representation. For $\lambda \neq 0$, which $G \subset SU(2) \times O(3)$ remains a symmetry of S?