Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4051-WE01

Title:
General Relativity IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 A two-dimensional spacetime has coordinates (t, x). New local coordinates are defined by $\rho = x^2$, $\kappa = t + x$.

(a) A vector field V^μ has components $V^\mu = (1, -1)$ with respect to the original coordinates. Calculate its components $\tilde{V}^\mu$ with respect to the new coordinates.

(b) The spacetime has metric $ds^2 = -dt^2 - 2dtdx$. Compute the components of the inverse metric in both coordinate systems.

Q2 The commutator of two vector fields is defined by

$$[U, V]^\mu = U^\nu \partial_\nu V^\mu - V^\nu \partial_\nu U^\mu .$$

(a) Show that the commutator is a vector field.

(b) In a spacetime with coordinates (t, x, y, z), U^μ has components $U^\mu = (t, x, y, z)$ and V^μ has components $V^\mu = (1, y, -x, 0)$. Calculate their commutator.

Q3 A scalar field has action

$$S[\phi] = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \phi^2 \right) ,$$

and stress tensor

$$T^{(\phi)}_{\mu\nu}(x) = \nabla_\mu \phi \nabla_\nu \phi + A g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi g_{\mu\nu} + B \phi^2 g_{\mu\nu} ,$$

where A and B are two constants.

(a) Compute the equation of motion of the scalar field. You can ignore boundary terms when integrating by parts.

(b) Fix the constants A and B by imposing that the stress tensor is conserved $\nabla_\mu T^{\mu\nu} = 0$ and using the equations of motion.

Q4 Consider the following black hole metric

$$ds^2 = -\left(1 - \frac{1}{r}\right)^2 dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) .$$

(a) What are the singularities? (No need to classify if they are coordinate or curvature singularities.)

(b) Can a light-ray sent at $r = 1/2$ go towards larger r? Justify your answers with a computation. If the answer is yes, what is the change in coordinate v as the light-ray goes between $r = 1/2$ and $r = 2/3$? If the answer is no, what is the change in coordinate v as the light-ray goes between $r = 2/3$ and $r = 1/2$? You can leave your answer as a definite integral.

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SECTION B

Q5 Consider a spacetime with metric

$$ds^2 = -2dudv + u^2dy^2 + dz^2$$

for $u > 0$.

(a) Show that on geodesics with $\dot{u} \neq 0$, we can take u as an affine parameter.

(b) Find the form of $v(u), y(u), z(u)$ for such geodesics.

(c) Given that $\omega_\mu = (1, 1, 1, 1)$ at $u = v = 1$, $y = z = 0$, and that ω_μ is parallel transported along the curve $u = v = 1$, $z = 0$, (that is, $U^\mu \nabla_\mu \omega_\nu = 0$, where U^μ is the tangent to the curve) find the value of ω_μ along the curve.

(d) Show that there are two-dimensional subspaces in this metric where the parallel transport is path independent.

Q6 Consider a spacetime with metric

$$ds^2 = dz^2 + f(z)^2(dx^2 + dy^2).$$

(a) Calculate the Christoffel symbols for this metric.

(b) Show that $U = \partial_z + x\partial_x + y\partial_y$ is a Killing vector for this metric for a particular choice of $f(z)$ which you should determine.

(c) The Ricci tensor for this metric is

$$R_{zz} = -2\frac{f''}{f}, \quad R_{xx} = R_{yy} = -aff'' - b(f')^2,$$

for some constants a, b. Using the contracted Bianchi identity, or otherwise, determine the values of a, b.

Q7 Consider the following cosmological metric

$$ds^2 = -dt^2 + (a(t))^2 \left(\frac{dr^2}{1-r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) .$$

The conservation and the Friedmann equations read respectively

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \,, \quad \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{8\pi G}{3}\rho \,,$$

where as usual overdots are derivatives with respect to coordinate time t.

(a) Is this an open or closed universe?

(b) A new type of matter obeys the equation of state

$$p = \rho \,.$$

Find how the density ρ depends on the scale factor $a(t)$ and describe the evolution of the scale factor of the universe. (Note you do not need an explicit solution for $a(t)$, only to justify its behaviour.)

(c) A spaceship moves along a path

$$r(t) = \alpha t + \frac{1}{\sqrt{2}} \,, \qquad \theta = \phi = 0 \,,$$

with $\alpha > 0$. At $t = 0$ find an upper bound for the value of α. Can the upper bound be attained?

(d) The spaceship of the previous part receives two light-rays one at $t = 0$ and another at $t = 3$. How much proper time elapses between receiving those two light-rays? You can leave your answer in terms of a definite integral.

Q8 Consider a following spacetime with metric

$$ds^2 = -r^2 d\eta^2 + dr^2.$$

(a) Does this metric have singularities? If so, where are the singularities? Justify your answer, but you do not need to decide if it is a coordinate or curvature singularity.

(b) Write down the equation for a timelike, spacelike and null geodesic in the form

$$\dot{r}^2 + V(r) = 0 \,,$$

where $V(r)$ is a function of r and of conserved quantities that you should determine.

(c) Can a massive observer following a geodesic be fixed at $r = 3$?

(d) Introduce new variables

$$x = r\cosh\eta \,, \qquad y = r\sinh\eta \,,$$

and obtain the metric in the new coordinates. If in this new metric you take x, $y \in \mathbb{R}$ can the old coordinates r,η cover all of this spacetime?