Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4061-WE01

Title:
Advanced Quantum Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Consider the theory of a complex scalar field $\phi(x)$ with the following action:

$$S = \int d^4x \left(-\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi\right)$$

(a) Write down the Euler-Lagrange equations of motion for ϕ and ϕ^*.

(b) Use Noether's theorem to construct the conserved current j^μ associated with the global $U(1)$ symmetry $\phi(x) \to e^{i\alpha}\phi(x)$.

(c) Consider the following modified action, where g is a real constant:

$$S' = \int d^4x \left(-\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi + g(\phi^3 + (\phi^*)^3)\right)$$

Explain why the current j^μ constructed in the previous part is no longer conserved on-shell. Write down an equation of the form $\partial_\mu j^\mu = g f(\phi, \phi^*)$ which holds on-shell, and where $f(\phi, \phi^*)$ is a function of the fields ϕ and ϕ^* that you should determine.

Q2 Consider the theory of a free real scalar field $\phi(x)$ with the following action:

$$S = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 \right)$$

The Hamiltonian may be written in terms of the usual creation and annihilation operators $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$ as

$$H = \int \frac{d^3k}{(2\pi)^3} \left(\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right) \qquad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$$

(In this expression we have omitted a normal-ordering constant as it will not be important for this problem). Denote the vacuum state of the free theory by $|0\rangle_0$. Here and throughout $\mathbf{k}$ denotes a 3-vector (k_x, k_y, k_z).

(a) Which of the following states are eigenstates of the Hamiltonian? If the state is an eigenstate then compute its eigenvalue.

- $|0\rangle_0$.
- $a_{\mathbf{k}}^\dagger |0\rangle_0$
- $a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2}^\dagger |0\rangle_0$
- $\left(a_{\mathbf{k}_1}^\dagger + a_{\mathbf{k}_2}^\dagger \right) |0\rangle_0$

(b) Does the free quantum field theory above have a state with energy eigenvalue $H = \frac{m}{2}$? If so, write down the state in terms of appropriate creation operators acting on the vacuum. If not, explain why it does not exist.

(c) Now consider the interacting quantum field theory with the following action:

$$S' = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \right)$$

Denote the vacuum of the interacting theory by $|0\rangle$. Is the state $a_{\mathbf{k}}^\dagger |0\rangle$ an eigenstate of the interacting Hamiltonian? Briefly explain your answer.

Q3 A scalar particle of type A and mass m_A decays into 2 identical scalar particles of type B and mass m_B (the decay happens at the same spacetime point), where $m_A > m_B$. The decay process takes place in 4 spacetime dimensions.

(a) Use the field theory language to write a Lagrangian of the system that describes the decay process, knowing that the interaction between the two types of particles has strength g.

Hint: There can be multiple fields and sources. You should be able to determine how many fields and sources you need.

(b) Write down an expression of the system's generating functional $Z[J]$ in the presence of the source(s) J that couple(s) to the field(s). Define the terms that appear in your expression.

(c) Express the system's generating functional $Z[J]$, that describes the decay process, as a formal functional derivative acting on $Z_0[J]$, where $Z_0[J]$ is the generating functional of the free field(s). Do not expand your expression, and do not take the functional derivatives.

(d) In an experiment, it was found that a single scalar particle of the type A discussed above simultaneously scatters with 3 identical massless scalar particles of a new type C. Write down a Lagrangian that describes this process, knowing that the strength of the interaction is λ and that the interaction happens at the same spacetime point.

Q4 (a) What is the difference between regularization and renormalization in field theory?

(b) Consider the following action of $(d+1)$-dimensional scalar field theory

$$S = -\int dx^{(d+1)} \left\{ \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi + \frac{\lambda}{4!}\phi^2\partial_\mu\phi\partial^\mu\phi \right\} .$$

Draw the one-loop Feynman diagram contributing to the four-point function

$$\langle 0|T\left\{ \hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) \right\}|0\rangle,$$

to the second order in λ.

Then, use the momentum-space techniques and introduce a cutoff scale Λ to compute this diagram in any dimension $(d+1)$.

(c) What is the degree of divergence of this diagram in spacetime dimension $(d+1)$?

SECTION B

Q5 Consider the interacting quantum field theory with the following action for a real scalar field $\phi(x)$:

$$S = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 \right)$$

(a) Write down the Feynman rules that are used to compute time-ordered correlation functions in this quantum field theory.

(b) Draw all of the Feynman diagrams that contribute to the following time-ordered correlation function in the interactive theory, up to order g^2.

$$\langle 0| T\left\{\phi(x_1)\phi(x_2)\right\} |0\rangle$$

For each diagram, write down its contribution to the two-point function. You may express your answer in terms of integrals over the position space Feynman propagator $G(x, y)$ – you do not need to perform these integrals.

(c) Do disconnected diagrams contribute to the correlation function above? Explain why (or why not).

(d) Compute the invariant matrix element $i\mathcal{M}$ up to order g^2 for the $2-2$ scattering of two ingoing ϕ particles with 3-momenta $\mathbf{p}_1$, $\mathbf{p}_2$ to two outgoing ϕ particles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$.

Q6 Consider the following theory of N scalar fields, $\phi_I(x)$, where $I, J, K \in \{1, \cdots N\}$ and $N > 1$:

$$S = \int d^4x \left(-\frac{1}{2} \sum_{I=1}^{N} \partial_\mu \phi_I \partial^\mu \phi_I - \frac{m^2}{2} \sum_{I=1}^{N} \phi_I^2 \right)$$

(a) Show that the infinitesimal transformation $\delta\phi_I = \sum_{J=1}^{N} \epsilon_{IJ}\phi_J$ is a symmetry of the action above, where $\epsilon_{IJ} = -\epsilon_{JI}$ is a constant antisymmetric $N \times N$ matrix. How many independent Noether currents do we expect from this symmetry?

(b) Show that the Noether currents for the above symmetry can be written as

$$j^\mu_{IJ} = \phi_I \partial^\mu \phi_J - \phi_J \partial^\mu \phi_I \ . \tag{1}$$

(c) We can write the field in terms of creation and annihilation operators $a^\dagger_{\mathbf{k},I}$, $a_{\mathbf{k},I}$ labeled by I as well as 3-momentum $\mathbf{k}$:

$$\phi_I(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k},I} e^{-i\omega_{\mathbf{k}} t + i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k},I} e^{i\omega_{\mathbf{k}} t - i\mathbf{k}\cdot\mathbf{x}} \right)$$

where the commutation relations are

$$[a^\dagger_{\mathbf{k},I}, a_{\mathbf{k}',J}] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')\delta_{IJ}$$

Write the conserved charge Q_{IJ} associated with the Noether current in Eq. (1) in terms of the creation and annihilation operators. You should express your answer in normal-ordered form.

(d) Use your answer to the previous part to compute the commutator

$$[Q_{IJ}, \phi_K(x)] \ .$$

Q7 Consider the interacting theory of two real scalar fields ϕ_1 and ϕ_2 with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_1^2 - \frac{\lambda}{4!}\phi_1^4 - \kappa\phi_1^2\phi_2^2.$$

(a) Write down an expression of the generating functional $Z[J_1, J_2]$ as a functional derivative of the free-field generating functional $Z_0[J_1, J_2]$ to a first order in the coupling constants λ and κ i.e., ignore terms of order λ^2, κ^2, $\lambda\kappa$, and higher. Here, J_1 and J_2 are two source currents that couple to ϕ_1 and ϕ_2, respectively.

(b) Carry out the functional derivatives of the interaction term using the diagrammatic technique (a cross for the external source, etc.) to find an expression of $Z[J_1, J_2]$ including terms that are first-order in λ and κ.

(c) Determine the normalization constant $\mathcal{N}$. Write the answer in terms of the free position Feynman propagators.

Q8 The following Lagrangian density describes a complex scalar field Φ, two left-handed Weyl fermions ψ_1 and ψ_2, as well as the interaction between the complex scalar and the fermions in 4 spacetime dimensions:

$$\mathcal{L} = -\partial_\mu \Phi \partial^\mu \Phi^* - V(\Phi, \Phi^*) + i\partial_\mu \bar{\psi}_1 \bar{\sigma}^\mu \psi_1 + i\partial_\mu \bar{\psi}_2 \bar{\sigma}^\mu \psi_2 - \frac{y}{2}(\Phi\psi_1\psi_2 + \Phi^* \bar{\psi}_1 \bar{\psi}_2)\,.$$

Here, y is a real parameter and the potential $V(\Phi, \Phi*)$ is given by

$$V(\Phi, \Phi*) = m^2 \Phi\Phi^* + \lambda(\Phi\Phi^*)^2\,,$$

and both m^2 and λ are real. Notice here that Φ^* is the complex conjugate of Φ.

(a) Show that the interaction term $\Phi\psi_1\psi_2 + \Phi^*\bar{\psi}_1\bar{\psi}_2$ is real.

(b) Show that the above Lagrangian density is invariant under the transformation:

$$\psi_1 \to e^{i\beta}\psi_1\,, \quad \psi_2 \to e^{i\beta}\psi_2\,, \quad \Phi \to e^{-2i\beta}\Phi\,,$$

where β is a constant real parameter.

(c) Assuming $m^2 < 0$, find the minimum (or minima) of the potential $V(\Phi, \Phi^*)$. What is the fermion mass at the minimum (or minima)?
Hint: Parametrise the complex field Φ as $\Phi = \rho e^{i\varphi}$.

(d) Find the degree of divergence of the two-point function $\langle 0|T\{\Phi(x)\Phi^*(y)\}|0\rangle$ to the second order in y.