

EXAMINATION PAPER

Examination Session: May/June

2024

Year:

Exam Code:

MATH40820-WE01

Title:

General Relativity V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



SECTION A

- Q1 A two-dimensional spacetime has coordinates (t, x). New local coordinates are defined by $\rho = x^2$, $\kappa = t + x$.
 - (a) A vector field V^{μ} has components $V^{\mu} = (1, -1)$ with respect to the original coordinates. Calculate its components \tilde{V}^{μ} with respect to the new coordinates.
 - (b) The spacetime has metric $ds^2 = -dt^2 2dtdx$. Compute the components of the inverse metric in both coordinate systems.
- Q2 The commutator of two vector fields is defined by

$$[U,V]^{\mu} = U^{\nu}\partial_{\nu}V^{\mu} - V^{\nu}\partial_{\nu}U^{\mu}.$$

- (a) Show that the commutator is a vector field.
- (b) In a spacetime with coordinates (t, x, y, z), U^{μ} has components $U^{\mu} = (t, x, y, z)$ and V^{μ} has components $V^{\mu} = (1, y, -x, 0)$. Calculate their commutator.

Q3 A scalar field has action

$$S[\phi] = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \phi^2 \right) \,,$$

and stress tensor

$$T^{(\phi)}_{\mu\nu}(x) = \nabla_{\mu}\phi\nabla_{\nu}\phi + Ag^{\alpha\beta}\nabla_{\alpha}\phi\nabla_{\beta}\phi g_{\mu\nu} + B\phi^2 g_{\mu\nu} \,,$$

where A and B are two constants.

- (a) Compute the equation of motion of the scalar field. You can ignore boundary terms when integrating by parts.
- (b) Fix the constants A and B by imposing that the stress tensor is conserved $\nabla_{\mu}T^{\mu\nu} = 0$ and using the equations of motion.

Q4 Consider the following black hole metric

$$ds^{2} = -\left(1 - \frac{1}{r}\right)^{2} dv^{2} + 2dvdr + r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2}).$$

- (a) What are the singularities? (No need to classify if they are coordinate or curvature singularities.)
- (b) Can a light-ray sent at r = 1/2 go towards larger r? Justify your answers with a computation. If the answer is yes, what is the change in coordinate v as the light-ray goes between r = 1/2 and r = 2/3? If the answer is no, what is the change in coordinate v as the light-ray goes between r = 2/3 and r = 1/2? You can leave your answer as a definite integral.



SECTION B

Q5 Consider a spacetime with metric

$$ds^2 = -2dudv + u^2dy^2 + dz^2$$

for u > 0.

- (a) Show that on geodesics with $\dot{u} \neq 0$, we can take u as an affine parameter.
- (b) Find the form of v(u), y(u), z(u) for such geodesics.
- (c) Given that $\omega_{\mu} = (1, 1, 1, 1)$ at u = v = 1, y = z = 0, and that ω_{μ} is parallel transported along the curve u = v = 1, z = 0, (that is, $U^{\mu}\nabla_{\mu}\omega_{\nu} = 0$, where U^{μ} is the tangent to the curve) find the value of ω_{μ} along the curve.
- (d) Show that there are two-dimensional subspaces in this metric where the parallel transport is path independent.

Q6 Consider a spacetime with metric

$$ds^{2} = dz^{2} + f(z)^{2}(dx^{2} + dy^{2}).$$

- (a) Calculate the Christoffel symbols for this metric.
- (b) Show that $U = \partial_z + x \partial_x + y \partial_y$ is a Killing vector for this metric for a particular choice of f(z) which you should determine.
- (c) The Ricci tensor for this metric is

$$R_{zz} = -2\frac{f''}{f}, \quad R_{xx} = R_{yy} = -aff'' - b(f')^2,$$

for some constants a, b. Using the contracted Bianchi identity, or otherwise, determine the values of a, b.

Q7 Consider the following cosmological metric

$$ds^{2} = -dt^{2} + (a(t))^{2} \left(\frac{dr^{2}}{1 - r^{2}} + r^{2} \left(d\theta^{2} + \sin^{2} \theta d\phi^{2} \right) \right) \,.$$

The conservation and the Friedmann equations read respectively

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \quad \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{8\pi G}{3}\rho,$$

where as usual overdots are derivatives with respect to coordinate time t.

- (a) Is this an open or closed universe?
- (b) A new type of matter obeys the equation of state

$$p = \rho$$
 .

Find how the density ρ depends on the scale factor a(t) and describe the evolution of the scale factor of the universe. (Note you do not need an explicit solution for a(t), only to justify its behaviour.)

(c) A spaceship moves along a path

$$r(t) = \alpha t + \frac{1}{\sqrt{2}}, \qquad \theta = \phi = 0,$$

with $\alpha > 0$. At t = 0 find an upper bound for the value of α . Can the upper bound be attained?

- (d) The spaceship of the previous part receives two light-rays one at t = 0 and another at t = 3. How much proper time elapses between receiving those two light-rays? You can leave your answer in terms of a definite integral.
- ${\bf Q8}\,$ Consider a following spacetime with metric

$$ds^2 = -r^2 d\eta^2 + dr^2.$$

- (a) Does this metric have singularities? If so, where are the singularities? Justify your answer, but you do not need to decide if it is a coordinate or curvature singularity.
- (b) Write down the equation for a timelike, spacelike and null geodesic in the form

$$\dot{r}^2 + V(r) = 0\,,$$

where V(r) is a function of r and of conserved quantities that you should determine.

- (c) Can a massive observer following a geodesic be fixed at r = 3?
- (d) Introduce new variables

$$x = r \cosh \eta$$
, $y = r \sinh \eta$,

and obtain the metric in the new coordinates. If in this new metric you take $x, y \in \mathbb{R}$ can the old coordinates r, η cover all of this spacetime?