Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH41220-WE01

Title:
Analysis V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Throughout this question, E is assumed to be a subset of the real numbers $\mathbb{R}$.

1.1 (a) State what it means for E to be countable.

(b) Give an example of a countable set E.

1.2 Show that if E is countable then E has Lebesgue outer measure equal to 0.

1.3 (a) State what it means for E to be Lebesgue measurable.

(b) Show that if E is countable then E is Lebesgue measurable.

(c) Give an example of a Lebesgue measurable E that is uncountable.

Q2 **2.1** Let $E \subset \mathbb{R}$ be measurable and let $1 \leq p < \infty$. State the definition of $L^p(E)$.

2.2 Consider $f : \mathbb{R} \to \mathbb{R}$, $f \in L^2(\mathbb{R})$. For $n \in \mathbb{N}$, we define

$$g_n(x) := \begin{cases} f(x), & \text{if } 0 \leq f(x) \leq n, \\ n, & \text{if } f(x) > n. \end{cases}$$

(a) Prove that $g_n \in L^2(\mathbb{R})$.

(b) Does $(g_n)_n$ converge to f in $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$? Give a full justification of your response.

Q3 **3.1** Let $(X, \|\cdot\|)$ be a normed linear space. State the definition of a bounded linear functional $T : X \to \mathbb{R}$.

3.2 Let $C^1[0,1]$ be the normed linear space of real-valued differentiable functions on $[0,1]$ with norm

$$\|f\| = \max_{x\in[0,1]} |f(x)|, \quad f \in C^1[0,1].$$

(a) Prove that $T_1 : C^1[0,1] \to \mathbb{R}$ defined by

$$T_1(f) := \int_0^1 f(x)\,dx$$

is a bounded linear functional on $C^1[0,1]$.

(b) Give an example of a linear functional $T_2 : C^1[0,1] \to \mathbb{R}$ which is not bounded. Provide a full justification of your response.

SECTION B

Q4 Recall that $\mathcal{M}$ denotes the measurable functions with domain $\mathbb{R}$ and taking values in the extended reals; and that $\mathcal{M}^+$ denotes the measurable functions with domain $\mathbb{R}$ and taking values in the nonnegative extended reals.

4.1 Let $f, h \in \mathcal{M}$. Suppose that h is integrable.

(a) Suppose that $|f| \leq h$. Explain why it follows that f is integrable.

(b) Show that h is finite almost everywhere.

4.2 State Fatou's Lemma.
(Hint: this is a statement concerning $\mathcal{M}^+$.)

4.3 Let $f_n \in \mathcal{M}$, for $n \in \mathbb{N}$. Suppose that $h \in \mathcal{M}$ is integrable and that $|f_n| \leq h$ for every $n \in \mathbb{N}$. *In the following, you may assume that* $\liminf_{n\to\infty} f_n \in \mathcal{M}$.

(a) Explain why $\liminf_{n\to\infty} f_n$ is integrable.

(b) Show that $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$.

Give full justifications of your responses.

Q5 **5.1** Let $f : \mathbb{R} \to \mathbb{R}$ and denote

$$f^+ : \mathbb{R} \to \mathbb{R}$$
$$f^+(x) = \max\{0, f(x)\}.$$

(a) State what it means for f to be (Lebesgue) measurable.

(b) Show that if f is (Lebesgue) measurable then the function f^+ is (Lebesgue) measurable. Fully justify your answer from the definition of measurability.

5.2 Let C be the middle-third Cantor set. Define the function $h : [0,1] \to \mathbb{R}$ by

$$h(x) = \begin{cases} \sin(1/x) & \text{if } x \in [0,1] \cap C \\ 0 & \text{if } x \notin C. \end{cases}$$

(a) Show that h is continuous on a set E with $[0,1] - E$ having Lebesgue measure 0.

(b) Is the function h Riemann integrable over $[0,1]$? Briefly justify your answer.

5.3 Assume that $r : \mathbb{R} \to \mathbb{R}$ is a function for which

$$2^{-1}|x-y| \leq |r(x) - r(y)| \leq 2|x-y|,$$

for all $x, y \in \mathbb{R}$. *In the following, you may use the fact that for any interval I, $r^{-1}(I)$ is an interval with length at most $2\ell(I)$.*

(a) Show that for any $E \subset \mathbb{R}$ we have $\mu^*(r^{-1}(E)) \leq 2\mu^*(E)$.

(b) Show that if E is Lebesgue measurable then $r^{-1}(E)$ is Lebesgue measurable. *You may use that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all open sets $U \subset \mathbb{R}$, $f^{-1}(U) \subset \mathbb{R}$ is open.*

(c) Show that for any $f \in \mathcal{M}^+$ we have $\int f \circ r \leq 2 \int f$. *You may use the identity $\mathbb{1}_E \circ r = \mathbb{1}_{r^{-1}(E)}$.*

Q6 Let $[a, b] \subset \mathbb{R}$ be a closed bounded interval. For $f \in L^1[a, b]$, consider

$$\|f\| = \int_{[a,b]} x^2 |f(x)|.$$

6.1 Is $\|\cdot\|$ a norm on $L^1[a, b]$? Justify your response.

6.2 Is $(L^1[a, b], \|\cdot\|)$ a Banach space? Justify your response.

6.3 Give an example of a normed linear space $(X, \|\cdot\|)$ such that there is a sequence of functions that is Cauchy in $(X, \|\cdot\|)$ but does not converge in $(X, \|\cdot\|)$. Briefly justify your response.

Q7 **7.1** Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\|\cdot\|$ denote the norm derived from the inner product. Let V be a finite dimensional subspace of H.

(a) State what it means for $\{\varphi_1, \ldots, \varphi_n\}$ to be an orthonormal set in $(V, \langle \cdot, \cdot \rangle)$.

(b) Now suppose $\{\varphi_1, \ldots, \varphi_n\}$ is an orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$. Show that for each $u \in H$, there exists a $w \in V$ such that

$$\|u - w\| = \min_{v \in V} \|u - v\|$$

by deriving an explicit formula for w.

7.2 Let G, F be infinite dimensional, separable Hilbert spaces. Prove that there exists a bijective linear transformation $T : F \to G$ such that

$$\|T(h)\|_G = \|h\|_F,$$

where $\|\cdot\|_G$, $\|\cdot\|_F$ are the norms derived from the inner products on G, F respectively. *You may use without proof that any separable Hilbert space has a countable orthonormal basis.*

SECTION C

Q8 (a) Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measure spaces. How is the σ-algebra $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ defined?

(b) State the Product Measure Theorem.

(c) Suppose that $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ are σ-finite measure spaces. Let $Z \in \mathcal{Z}$. Prove that for any nonnegative measurable function f on Z, we have

$$\int_X \left(\int_Y f \, d\nu \right) d\mu = \int_Y \left(\int_X f \, d\mu \right) d\nu.$$

You may use the fact that for $\psi = \mathbb{1}_E$ where $E \in \mathcal{Z}$, we have

$$\int_X \left(\int_Y \psi \, d\nu \right) d\mu = \int_Y \left(\int_X \psi \, d\mu \right) d\nu.$$