

EXAMINATION PAPER

Examination Session: May/June

2024

Year:

Exam Code:

MATH41220-WE01

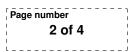
Title:

Analysis V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



SECTION A

- **Q1** Throughout this question, E is assumed to be a subset of the real numbers \mathbb{R} .
 - 1.1 (a) State what it means for E to be countable.(b) Give an example of a countable set E.
 - **1.2** Show that if E is countable then E has Lebesgue outer measure equal to 0.
 - **1.3** (a) State what it means for E to be Lebesgue measurable.
 - (b) Show that if E is countable then E is Lebesgue measurable.
 - (c) Give an example of a Lebesgue measurable E that is uncountable.
- Q2 2.1 Let $E \subset \mathbb{R}$ be measurable and let $1 \leq p < \infty$. State the definition of $L^p(E)$. 2.2 Consider $f : \mathbb{R} \to \mathbb{R}, f \in L^2(\mathbb{R})$. For $n \in \mathbb{N}$, we define

$$g_n(x) := \begin{cases} f(x), & \text{if } 0 \le f(x) \le n, \\ n, & \text{if } f(x) > n. \end{cases}$$

- (a) Prove that $g_n \in L^2(\mathbb{R})$.
- (b) Does $(g_n)_n$ converge to f in $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$? Give a full justification of your response.
- **Q3** 3.1 Let $(X, \|\cdot\|)$ be a normed linear space. State the definition of a bounded linear functional $T: X \to \mathbb{R}$.
 - **3.2** Let $C^{1}[0,1]$ be the normed linear space of real-valued differentiable functions on [0,1] with norm

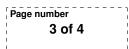
$$||f|| = \max_{x \in [0,1]} |f(x)|, \quad f \in C^1[0,1].$$

(a) Prove that $T_1: C^1[0,1] \to \mathbb{R}$ defined by

$$T_1(f) := \int_0^1 f(x) \, dx$$

is a bounded linear functional on $C^{1}[0, 1]$.

(b) Give an example of a linear functional $T_2 : C^1[0,1] \to \mathbb{R}$ which is not bounded. Provide a full justification of your response.



SECTION B

- Q4 Recall that \mathcal{M} denotes the measurable functions with domain \mathbb{R} and taking values in the extended reals; and that \mathcal{M}^+ denotes the measurable functions with domain \mathbb{R} and taking values in the nonnegative extended reals.
 - **4.1** Let $f, h \in \mathcal{M}$. Suppose that h is integrable.
 - (a) Suppose that $|f| \leq h$. Explain why it follows that f is integrable.
 - (b) Show that h is finite almost everywhere.
 - **4.2** State Fatou's Lemma. (*Hint: this is a statement concerning* \mathcal{M}^+ .)
 - **4.3** Let $f_n \in \mathcal{M}$, for $n \in \mathbb{N}$. Suppose that $h \in \mathcal{M}$ is integrable and that $|f_n| \leq h$ for every $n \in \mathbb{N}$. In the following, you may assume that $\liminf_{n\to\infty} f_n \in \mathcal{M}$.
 - (a) Explain why $\liminf_{n\to\infty} f_n$ is integrable.
 - (b) Show that $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$.

Give full justifications of your responses.

Q5 5.1 Let $f : \mathbb{R} \to \mathbb{R}$ and denote

$$f^+:\mathbb{R}\to\mathbb{R}$$

$$f^+(x) = \max\{0, f(x)\}.$$

- (a) State what it means for f to be (Lebesgue) measurable.
- (b) Show that if f is (Lebesgue) measurable then the function f^+ is (Lebesgue) measurable. Fully justify your answer from the definition of measurability.
- **5.2** Let C be the middle-third Cantor set. Define the function $h: [0,1] \to \mathbb{R}$ by

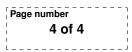
$$h(x) = \begin{cases} \sin(1/x) & \text{if } x \in [0,1] \cap C \\ 0 & \text{if } x \notin C. \end{cases}$$

- (a) Show that h is continuous on a set E with [0,1] E having Lebesgue measure 0.
- (b) Is the function h Riemann integrable over [0,1]? Briefly justify your answer.
- **5.3** Assume that $r : \mathbb{R} \to \mathbb{R}$ is a function for which

$$2^{-1}|x-y| \le |r(x) - r(y)| \le 2|x-y|,$$

for all $x, y \in \mathbb{R}$. In the following, you may use the fact that for any interval I, $r^{-1}(I)$ is an interval with length at most $2\ell(I)$.

- (a) Show that for any $E \subset \mathbb{R}$ we have $\mu^*(r^{-1}(E)) \leq 2\mu^*(E)$.
- (b) Show that if E is Lebesgue measurable then $r^{-1}(E)$ is Lebesgue measurable. You may use that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all open sets $U \subset \mathbb{R}$, $f^{-1}(U) \subset \mathbb{R}$ is open.
- (c) Show that for any $f \in \mathcal{M}^+$ we have $\int f \circ r \leq 2 \int f$. You may use the identity $\mathbb{1}_E \circ r = \mathbb{1}_{r^{-1}(E)}$.





Q6 Let $[a, b] \subset \mathbb{R}$ be a closed bounded interval. For $f \in L^1[a, b]$, consider

$$||f|| = \int_{[a,b]} x^2 |f(x)|.$$

- **6.1** Is $\|\cdot\|$ a norm on $L^1[a, b]$? Justify your response.
- **6.2** Is $(L^1[a, b], \|\cdot\|)$ a Banach space? Justify your response.
- **6.3** Give an example of a normed linear space $(X, \|\cdot\|)$ such that there is a sequence of functions that is Cauchy in $(X, \|\cdot\|)$ but does not converge in $(X, \|\cdot\|)$. Briefly justify your response.
- **Q7** 7.1 Let *H* be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\|\cdot\|$ denote the norm derived from the inner product. Let *V* be a finite dimensional subspace of *H*.
 - (a) State what it means for $\{\varphi_1, \ldots, \varphi_n\}$ to be an orthonormal set in $(V, \langle \cdot, \cdot \rangle)$.
 - (b) Now suppose $\{\varphi_1, \ldots, \varphi_n\}$ is an orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$. Show that for each $u \in H$, there exists a $w \in V$ such that

$$||u - w|| = \min_{v \in V} ||u - v||$$

by deriving an explicit formula for w.

7.2 Let G, F be infinite dimensional, separable Hilbert spaces. Prove that there exists a bijective linear transformation $T: F \to G$ such that

$$||T(h)||_G = ||h||_F,$$

where $\|\cdot\|_G$, $\|\cdot\|_F$ are the norms derived from the inner products on G, F respectively. You may use without proof that any separable Hilbert space has a countable orthonormal basis.

SECTION C

- **Q8** (a) Let (X, \mathcal{X}) and (Y, \mathcal{Y}) be measure spaces. How is the σ -algebra $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ defined?
 - (b) State the Product Measure Theorem.
 - (c) Suppose that (X, \mathcal{X}, μ) and (Y, \mathcal{Y}, ν) are σ -finite measure spaces. Let $Z \in \mathcal{Z}$. Prove that for any nonnegative measurable function f on Z, we have

$$\int_X \left(\int_Y f \, d\nu \right) d\mu = \int_Y \left(\int_X f \, d\mu \right) d\nu$$

You may use the fact that for $\psi = \mathbb{1}_E$ where $E \in \mathbb{Z}$, we have

$$\int_X \left(\int_Y \psi \, d\nu \right) d\mu = \int_Y \left(\int_X \psi \, d\mu \right) d\nu.$$