Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH41320-WE01

Title:
Riemannian Geometry V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Prove or disprove the following assertions:

1.1 *The subset of* $\mathbb{R}^3$ *given by*

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid 4x^2 + 6y^2 + 3z^2 = 2\}$$

is a smooth 2*-dimensional submanifold of* $\mathbb{R}^3$.

1.2 *The subset of* $\mathbb{R}^3$ *given by*

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid (x-1)^2+y^2+z^2 = 1\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid (x+1)^2+y^2+z^2 = 1\}$$

is a smooth submanifold of $\mathbb{R}^3$.

Q2 **2.1** State the definition of the tangent bundle TM of a differentiable manifold M.

2.2 Let M be an n-dimensional differentiable manifold. Show that TM is $2n$-dimensional. You may assume that TM is a differentiable manifold.

2.3 Let

$$X(x, y, z) = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y + 1)\frac{\partial}{\partial z},$$
$$Y(x, y, z) = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.$$

be two differentiable vector fields on $\mathbb{R}^3$ equipped with the usual chart given by the identity. Compute the Lie bracket $[X, Y]$.

Q3 Prove or disprove the following assertions:

3.1 *The smooth manifolds* $S^2 \times S^2$ *and* $S^2 \times S^3$ *are diffeomorphic.*

3.2 *There exists a Riemannian metric* g *on* $\mathbb{R}^2$ *such that* $(g_{ij}(p))$*, the expression in local coordinates of* g *at some point* $p \in \mathbb{R}^2$ *for some given coordinate chart around* p*, is the matrix*

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

3.3 *The smooth manifold* $S^1 \times \mathbb{R}$ *admits a complete Riemannian metric with sectional curvature* $K \geq 1$.

Q4 Let (M, g) be a Riemannian manifold.

4.1 Prove or disprove the following assertion: *The Levi–Civita connection is a* $\binom{2}{1}$ *tensor field on* M.

4.2 State the Jacobi equation for a differentiable vector field V along a geodesic $c\colon [a, b] \to M$.

4.3 Let $J\colon [a, b] \to TM$ be a Jacobi field along a geodesic $c\colon [a, b] \to M$ and let $f(t) = ||J(t)||^2$. Show that if M has non-positive sectional curvature, then $f''(t) \geq 0$.

SECTION B

Q5 **5.1** Let (M, g) be an n-dimensional complete Riemannian manifold, fix $p \in M$ and suppose that $\exp_p \colon T_pM \to M$ is a diffeomorphism. Let $\varphi \colon T_pM \to T_pM$ be an orthogonal transformation and define $f \colon M \to M$ by $f = \exp_p \circ \varphi \circ \exp_p^{-1}$. Show that Df_p is a linear isomorphism of T_pM given by $Df_p = \psi \circ \varphi \circ \psi^{-1}$, where ψ is a linear isomorphism of T_pM.

5.2 Let (M, g) be a Riemannian manifold and let $f \colon M \to \mathbb{R}$ be a smooth function. The *gradient* of f, denoted by $\mathrm{grad}f$, is the smooth vector field on M characterised by

$$Y(f) = \langle \mathrm{grad}f, Y \rangle$$

for all $Y \in \mathcal{X}(M)$. Show that, if X and Y are smooth vector fields on M, then

$$\langle \nabla_X \mathrm{grad}f, Y \rangle = \langle \nabla_Y \mathrm{grad}f, X \rangle.$$

You may use in your computations the fact that the Levi–Civita connection is torsion-free and compatible with the Riemannian metric, as well as the properties of the Lie bracket.

5.3 Let (M, g) be a Riemannian manifold and let $f \colon M \to \mathbb{R}$ be a smooth function. Consider the smooth vector field X on M given by $X(p) = \mathrm{grad}f(p)$ for all $p \in M$. An *integral curve* of X through $p_o \in M$ is a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p_o$ and $\gamma'(t) = X(\gamma(t))$ for all $t \in (-\varepsilon, \varepsilon)$. Use the identity in question **5.2** to show that, if $||\mathrm{grad}f(p)|| = c > 0$ for all $p \in M$, then the integral curves of X are geodesics.

Q6 Consider the smooth manifold $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ with the global chart given by the identity map. Let g be the Riemannian metric on M given by $g_{11}(x, y) = 1$, $g_{12}(x, y) = g_{21}(x, y) = 0$, and $g_{22}(x, y) = e^{2x}$.

6.1 Compute the sectional curvature of M.

6.2 Let $a, b, c > 0$. Show that the differentiable curve $\gamma \colon (0, \infty) \to M$ given by $\gamma(t) = (at + b, c)$ is a geodesic in M.

Q7 Fix a positive integer n and let $M(2n, \mathbb{R})$ denote the set of $2n \times 2n$ real matrices. Let Id_n be the $n \times n$ identity matrix and define

$$\mathrm{Sp}(2n) = \{A \in M(2n, \mathbb{R}) \mid A^T \Omega A = \Omega\},$$

where

$$\Omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}.$$

The elements of $\mathrm{Sp}(2n)$ are called $2n \times 2n$ *symplectic matrices.*

7.1 Show that $\mathrm{Sp}(2n)$ is a group under matrix multiplication.

7.2 Show that $\mathrm{Sp}(2)$ is a Lie group and determine its dimension.

7.3 Express $T_{\mathrm{Id}_2}\mathrm{Sp}(2)$ as a subset of $M(2, \mathbb{R})$.

Q8 **8.1** State the definition of a local isometry between two Riemannian manifolds (M, g) and (N, h).

8.2 Let $\widetilde{M}$ and M be smooth connected manifolds. A map $\pi\colon \widetilde{M} \to M$ is a *covering map* if π is smooth and surjective, and each point in M has a neighbourhood U such that π maps each component of $\pi^{-1}(U)$ diffeomorphically onto U. Suppose that M has a Riemannian metric g. Show that $\widetilde{M}$ has a Riemannian metric $\widetilde{g}$ such that the covering map $\pi\colon \widetilde{M} \to M$ is a local isometry.

8.3 Let M be a differentiable manifold and let ∇ be an affine connection on M. Show that the map $T\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

defines a $\binom{2}{1}$ tensor field on M. You may use without proof the properties of an affine connection and of the Lie bracket.