Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH41420-WE01

Title:
Solitons V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Find a travelling wave solution to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

with boundary conditions u, u_x, $u_{xx} \to 0$ as $x \to \pm\infty$. You can use the indefinite integral

$$\int \frac{df}{f\sqrt{1-f}} = -2\,\mathrm{arcsech}(\sqrt{f}) + \mathrm{const}$$

without proof.

Q2 The Marchenko equation is the equation

$$K(x,z;t) + F(x+z;t) + \int_{-\infty}^{x} dy\ K(x,y;t)F(y+z;t) = 0$$

for the unknown $K(x,z;t)$, and with t a real parameter. If $F(x;t) = \frac{1}{2}e^{x/2-t}$, find a solution $K(x,z;t)$ of the Marchenko equation of the form

$$K(x,z;t) = g(x,t)e^{z/2}\ .$$

Using

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x;t)\ ,$$

show that

$$u(x,t) = a\ \mathrm{sech}^2(bx+ct)$$

for values of the constants a, b, c that you should find.

Q3 The motion of a rigid body freely rotating about its centre of mass in the absence of gravity is described by the equation of motion $\frac{d}{dt}\boldsymbol{\ell} = \boldsymbol{\ell} \times \boldsymbol{\omega}$, where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3)$ is the angular momentum and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the rigid body.

3.1 Show that this equation of motion can be written in the Lax form $\frac{d}{dt}L = [M, L]$ where

$$L = \begin{pmatrix} 0 & \ell_3 & -\ell_2 \\ -\ell_3 & 0 & \ell_1 \\ \ell_2 & -\ell_1 & 0 \end{pmatrix}\ , \qquad M = c\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

for a value of the constant c that you should find.

3.2 Use the Lax form of the equation to show that $\mathrm{tr}(L)$, $\mathrm{tr}(L^2)$ and $\mathrm{tr}(L^3)$ are conserved, and find these quantities explicitly.

SECTION B

Q4 **4.1** If $u(x,t)$ is *any* solution of the KdV equation considered in Q1, show that $\tilde{u}(x,t) = u(x+at,t)+b$ is a solution to the same equation, provided the constants a and b are related to each other in a way that you should determine. Using this fact and your answer to Q1, or otherwise, find a travelling wave solution of the KdV equation with modified boundary conditions $u \to C$, u_x, $u_{xx} \to 0$ as $x \to \pm\infty$, where C is a constant (with the same value at $+\infty$ and $-\infty$).

4.2 We now seek a travelling wave solution to the mKdV equation

$$w_t - 6w^2 w_x + w_{xxx} = 0$$

with kink-like boundary conditions $w \to -D$ as $x \to -\infty$, $w \to +D$ as $x \to +\infty$, and w_x, $w_{xx} \to 0$ as $x \to \pm\infty$, where D is a nonzero constant.

(i) Show that the velocity v of the travelling wave must be equal to kD^2, where k is a constant you should find.

(ii) Find the travelling wave. You can use the indefinite integral

$$\int \frac{df}{1-f^2} = \operatorname{arctanh}(f) + \text{const}$$

without proof.

4.3 Show that the Miura transformation $u = -w^2 - w_x$ of the solution you found in part 4.2(ii) is equal either to a constant, or to one of the solutions you found in part 4.1, depending on the sign of D.

Q5 **5.1** A field $u(x,t)$ is defined on the infinite line $-\infty < x < \infty$. Its energy is given by

$$E[u] = \int_{-\infty}^{\infty} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}(u^2-1)^2\,dx$$

and it satisfies the 'kink' boundary conditions u_t, $u_x \to 0$ as $x \to \pm\infty$, $u \to -1$ as $x \to -\infty$, $u \to +1$ as $x \to +\infty$.

Use the Bogomol'nyi argument to show that $E[u] \geq K$, where K is a positive constant which you should determine, and find all solutions u which saturate this bound. The indefinite integral given in Q4 can be used without proof.

5.2 The field $u(x,t)$ is now confined to the interval $0 \leq x \leq a$, where a is a positive constant, and the boundary conditions $u(0,t) = 0$, $u(a,t) = \frac{1}{2}$ are imposed. The energy has the same form as in part 6.1, but with the integral now running from 0 to a:

$$E[u] = \int_0^a \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}(u^2-1)^2\,dx\,.$$

Adapt your argument from part 5.1 to show that this energy satisfies $E[u] \geq K'$, where K' is another positive constant which you should determine. For what value of a is it possible for this bound to be saturated?

Q6 Consider the functional

$$F[u] = \int_{-\infty}^{+\infty} dx \; f(u, u_x, u_{xx})$$

of a field u which satisfies the boundary conditions $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$.

6.1 Derive an expression for the functional derivative $\delta F[u]/\delta u$ in terms of the partial derivatives of $f(u, u_x, u_{xx})$.

6.2 Find a functional $F[u]$ of the above form such that the equation

$$u_t = \frac{\partial}{\partial x} \frac{\delta F[u]}{\delta u}$$

is the same as the partial differential equation

$$u_t + u_{xxxxx} + 20 u_x u_{xx} + 10 u_x + 10 u u_{xxx} + 30 u^2 u_x = 0 \; .$$

Q7 **7.1** Let $M = \frac{d}{dx} + \frac{1}{x}$. Show that

$$M M^\dagger = -\frac{d^2}{dx^2} \; , \quad M^\dagger M = -\frac{d^2}{dx^2} + \frac{2}{x^2} \; .$$

Describe how the eigenfunction ψ of the equation

$$M^\dagger M \psi = E \psi$$

can be related to the eigenfunction χ of the equation

$$M M^\dagger \chi = E \chi \; ,$$

and use this relation to find ψ explicitly when $E = k^2 > 0$.

7.2 Use the previous results to find the reflection and transmission coefficients $R(k)$ and $T(k)$ for the scattering problem with the potential

$$V(x) = \begin{cases} 2(x+1)^{-2} \; , & x \geq 0 \\ 0 \; , & x < 0 \end{cases} \; .$$

SECTION C

Q8 A stationary breather solution of the sine-Gordon equation on the real line has the form

$$\tan \frac{u(x,t)}{4} = \cot \varphi \cdot \frac{\sin(\sin \varphi \cdot t)}{\cosh(\cos \varphi \cdot x)} \; ,$$

where φ is a constant angle with $0 < \varphi < \pi/2$. What is the period τ of this solution? Show that when $\varphi \ll 1$, $\tau \sim A/\varphi$, while $x_{\max} \sim B \log \varphi$, where $x_{\max}$ is the maximal spatial size of the breather, and A and B are constants you should find. (For the purpose of this question we define $x_{\max}$ to be the value of $x > 0$ for which $\tan(u/4) = 1$ when $t = \tau/4$ and the oscillatory factor in the numerator is at its maximum.)