

## **EXAMINATION PAPER**

Examination Session: May/June

2024

Year:

Exam Code:

MATH41420-WE01

Title:

Solitons V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

**Revision:** 

## SECTION A

Q1 Find a travelling wave solution to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

with boundary conditions  $u, u_x, u_{xx} \to 0$  as  $x \to \pm \infty$ . You can use the indefinite integral

$$\int \frac{df}{f\sqrt{1-f}} = -2\operatorname{arcsech}(\sqrt{f}) + \operatorname{const}$$

without proof.

**Q2** The Marchenko equation is the equation

$$K(x, z; t) + F(x + z; t) + \int_{-\infty}^{x} dy \ K(x, y; t) F(y + z; t) = 0$$

for the unknown K(x, z; t), and with t a real parameter. If  $F(x; t) = \frac{1}{2}e^{x/2-t}$ , find a solution K(x, z; t) of the Marchenko equation of the form

$$K(x,z;t) = g(x,t)e^{z/2} .$$

Using

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x;t) ,$$

show that

$$u(x,t) = a \operatorname{sech}^2(bx + ct)$$

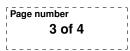
for values of the constants a, b, c that you should find.

- **Q3** The motion of a rigid body freely rotating about its centre of mass in the absence of gravity is described by the equation of motion  $\frac{d}{dt}\boldsymbol{\ell} = \boldsymbol{\ell} \times \boldsymbol{\omega}$ , where  $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3)$  is the angular momentum and  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$  is the angular velocity of the rigid body.
  - **3.1** Show that this equation of motion can be written in the Lax form  $\frac{d}{dt}L = [M, L]$  where

$$L = \begin{pmatrix} 0 & \ell_3 & -\ell_2 \\ -\ell_3 & 0 & \ell_1 \\ \ell_2 & -\ell_1 & 0 \end{pmatrix} , \qquad M = c \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

for a value of the constant c that you should find.

**3.2** Use the Lax form of the equation to show that tr(L),  $tr(L^2)$  and  $tr(L^3)$  are conserved, and find these quantities explicitly.



## SECTION B

- **Q4** 4.1 If u(x,t) is any solution of the KdV equation considered in Q1, show that  $\tilde{u}(x,t) = u(x + at, t) + b$  is a solution to the same equation, provided the constants a and b are related to each other in a way that you should determine. Using this fact and your answer to Q1, or otherwise, find a travelling wave solution of the KdV equation with modified boundary conditions  $u \to C$ ,  $u_x$ ,  $u_{xx} \to 0$  as  $x \to \pm \infty$ , where C is a constant (with the same value at  $+\infty$  and  $-\infty$ ).
  - 4.2 We now seek a travelling wave solution to the mKdV equation

$$w_t - 6w^2w_x + w_{xxx} = 0$$

with kink-like boundary conditions  $w \to -D$  as  $x \to -\infty$ ,  $w \to +D$  as  $x \to +\infty$ , and  $w_x, w_{xx} \to 0$  as  $x \to \pm\infty$ , where D is a nonzero constant.

- (i) Show that the velocity v of the travelling wave must be equal to  $kD^2$ , where k is a constant you should find.
- (ii) Find the travelling wave. You can use the indefinite integral

$$\int \frac{df}{1-f^2} = \operatorname{arctanh}(f) + \operatorname{const}$$

without proof.

- **4.3** Show that the Miura transformation  $u = -w^2 w_x$  of the solution you found in part 4.2(ii) is equal either to a constant, or to one of the solutions you found in part 4.1, depending on the sign of D.
- **Q5** 5.1 A field u(x,t) is defined on the infinite line  $-\infty < x < \infty$ . Its energy is given by

$$E[u] = \int_{-\infty}^{\infty} \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}(u^2 - 1)^2 dx$$

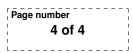
and it satisfies the 'kink' boundary conditions  $u_t, u_x \to 0$  as  $x \to \pm \infty, u \to -1$  as  $x \to -\infty, u \to +1$  as  $x \to +\infty$ .

Use the Bogomol'nyi argument to show that  $E[u] \ge K$ , where K is a positive constant which you should determine, and find all solutions u which saturate this bound. The indefinite integral given in Q4 can be used without proof.

**5.2** The field u(x,t) is now confined to the interval  $0 \le x \le a$ , where a is a positive constant, and the boundary conditions u(0,t) = 0,  $u(a,t) = \frac{1}{2}$  are imposed. The energy has the same form as in part 6.1, but with the integral now running from 0 to a:

$$E[u] = \int_0^a \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}(u^2 - 1)^2 dx$$

Adapt your argument from part 5.1 to show that this energy satisfies  $E[u] \ge K'$ , where K' is another positive constant which you should determine. For what value of a is it possible for this bound to be saturated?



Q6 Consider the functional

$$F[u] = \int_{-\infty}^{+\infty} dx \ f(u, u_x, u_{xx})$$

of a field u which satisfies the boundary conditions  $u, u_x, u_{xx} \to 0$  as  $|x| \to \infty$ .

- **6.1** Derive an expression for the functional derivative  $\delta F[u]/\delta u$  in terms of the partial derivatives of  $f(u, u_x, u_{xx})$ .
- **6.2** Find a functional F[u] of the above form such that the equation

$$u_t = \frac{\partial}{\partial x} \frac{\delta F[u]}{\delta u}$$

is the same as the partial differential equation

$$u_t + u_{xxxxx} + 20u_x u_{xx} + 10u_x + 10u u_{xxx} + 30u^2 u_x = 0 .$$

**Q7** 7.1 Let  $M = \frac{d}{dx} + \frac{1}{x}$ . Show that

$$M M^{\dagger} = - \frac{d^2}{dx^2} \ , \quad M^{\dagger} M = - \frac{d^2}{dx^2} + \frac{2}{x^2} \ .$$

Describe how the eigenfunction  $\psi$  of the equation

$$M^{\dagger}M\psi = E\psi$$

can be related to the eigenfunction  $\chi$  of the equation

$$M M^{\dagger} \chi = E \chi \; ,$$

and use this relation to find  $\psi$  explicitly when  $E = k^2 > 0$ .

**7.2** Use the previous results to find the reflection and transmission coefficients R(k) and T(k) for the scattering problem with the potential

$$V(x) = \begin{cases} 2(x+1)^{-2} , & x \ge 0\\ 0 , & x < 0 \end{cases}$$

## SECTION C

 $\mathbf{Q8}$  A stationary breather solution of the sine-Gordon equation on the real line has the form

$$\tan \frac{u(x,t)}{4} = \cot \varphi \cdot \frac{\sin(\sin \varphi \cdot t)}{\cosh(\cos \varphi \cdot x)} ,$$

where  $\varphi$  is a constant angle with  $0 < \varphi < \pi/2$ . What is the period  $\tau$  of this solution? Show that when  $\varphi \ll 1$ ,  $\tau \sim A/\varphi$ , while  $x_{\max} \sim B \log \varphi$ , where  $x_{\max}$  is the maximal spatial size of the breather, and A and B are constants you should find. (For the purpose of this question we define  $x_{\max}$  to be the value of x > 0 for which  $\tan(u/4) = 1$  when  $t = \tau/4$  and the oscillatory factor in the numerator is at its maximum.)