EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH41520-WE01

Title:

Topics in Algebra and Geometry V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

SECTION A

Q1 (a) Formulate what it means for a function $f : \mathbb{H} \to \mathbb{C}$ to be weakly modular for some subgroup Γ of the modular group Γ_1.

(b) Let $N_1,\, N_2 \in \mathbb{Z}_{>0}$ and $f(\tau) \in M_k(\Gamma_0(N_1))$ for some positive integer weight k. Show that the function $g : \mathbb{H} \to \mathbb{C}$ defined by

$$g(\tau) = f(N_2\tau) \qquad \forall \tau \in \mathbb{H}$$

is weakly modular for $\Gamma_0(N_1N_2)$.

(c) Give the description of the space of modular forms for Γ_1 as a polynomial ring in finitely many generators. Write the Eisenstein series $E_{14}(\tau)$ and the square $\Delta(\tau)^2$ of the discriminant function as a polynomial in those generators. Justify your answer.

Q2 (a) Let $\mathcal{F}$ be the standard fundamental domain for the action of Γ_1 on $\mathbb{H}$. Describe its elements using suitable inequalities (alternatively, you may want to describe its boundary as a union of curve segments).

(b) State the valency formula pertaining to meromorphic modular forms for Γ_1, and deduce from it, with proof, an upper bound on $\dim_{\mathbb{C}} M_k(\Gamma_1)$ for even k.

(c) Give an example of a weakly holomorphic modular form for Γ_1 of weight -20. (Recall that a weakly holomorphic modular form is only allowed to have poles at infinity.)

Q3 Let λ be an arithmetic function defined by

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ is a product of potentially non-distinct primes } p_i, \end{cases}$$

and let $\mathbf{1}$ denote the constant function

$$\mathbf{1}(n) := 1.$$

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Define an arithmetic function f by

$$f(n) = \lfloor \sqrt{n} \rfloor - \lfloor \sqrt{n-1} \rfloor.$$

(a) Prove that $f(n) = 1$ if n is a perfect square, and $f(n) = 0$ otherwise.

(b) Is $f(x)$ multiplicative? Is it completely multiplicative? Give a thorough reasoning.

(c) Show that $f = \lambda * \mathbf{1}$.

Q4 Here, and throughout this exam let

$$\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2} dt \tag{1}$$

denote the *Euler–Mascheroni constant*, where $\{t\}$ denotes the fractional part of a real number t.

(a) Prove that for $x \geq 1$,

$$\sum_{1 \leq n \leq x} \frac{1}{n} = \log x + \gamma + O(1/x).$$

(b) Using the Dirichlet hyperbola method (or otherwise), prove that

$$\sum_{1 \leq n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}),$$

where $d(n) := \sum_{k|n} 1$ denotes the divisor function.

SECTION B

Q5 Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$, $f \in M_k(\Gamma_1)$ and $g \in M_\ell(\Gamma_1)$ for $k,\ \ell \in \mathbb{Z}_{>0}$. Denote the automorphy factor of $\gamma \in \Gamma_1$ with respect to $\tau \in \mathbb{H}$ by $j(\gamma, \tau)$.

(a) Compute $\frac{d}{d\tau}(\gamma.\tau)$.

(b) By computing $\frac{d}{d\tau} f(\gamma.\tau)$ in two different ways (on the one hand, apply the chain rule, whereas on the other hand you may first want to apply the modular property), express $f'(\gamma\tau)$ in terms of $j(\gamma, \tau)$, $f(\tau)$, and $f'(\tau)$, where f' denotes the derivative with respect to the argument. (Think of γ as a map from $\mathbb{H}$ to itself. You probably want to take (a) into account.)

(c) Show that the combination

$$[f, g] := [f, g]_{k,\ell} := kfg' - \ell f'g$$

is itself modular for Γ_1, of weight $w(k, \ell) = k + \ell + 2$.

(d) Now consider a third modular form $h \in M_m(\Gamma_1)$ ($m \in \mathbb{Z}_{>0}$).
Note that, in view of the previous part, we can now form the nested bracket $[[f, g], h] := [[f, g]_{k,\ell}, h]_{w(k,\ell),m}$, defining another modular form, and similarly $[[g, h], f]$ and $[[h, f], g]$ (where the appropriate weight subscripts have been suppressed for ease of reading).
Show that we have the so-called *Jacobi identity*

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0\,.$$

(Hint: Notice the cyclic symmetry that should help to cut down calculations.)

Q6 (a) For a prime p, write down the explicit formula for the p-th Hecke operator T_p mapping the vector space $M_k(\Gamma_1)$ to a space of functions from $\mathbb{H}$ to $\mathbb{C}$.

(b) Show that the function T_pf, for $f \in M_k(\Gamma_1)$, is periodic of period 1, that is to say

$$T_pf(\tau + 1) = T_pf(\tau) \qquad \forall \tau \in \mathbb{H}.$$

(c) Give a formula, together with a proof, for the Fourier expansion of T_pf around the cusp at infinity in terms of the coefficients a_n and suitable powers of p in $f(\tau) = \sum_{n\geq 0} a_n q^n$ (with $q = e^{2\pi i\tau}$) where $f \in M_k(\Gamma_1)$.

(d) For a fixed prime p, give a recursive formula for T_{p^n} ($n \geq 0$), discussing the case $n = 0$ first.

(e) Compute the eigenvalues of T_{2^r} applied to Ramanujan's discriminant function $\Delta(\tau)$, for $r = 1$, 2, and 3. Justify your answer.

Q7 (a) Prove that for all complex numbers s with $\mathrm{Re}(s) > 1$, we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{t\}t^{-s-1}\,dt.$$

(b) Prove that $(s-1)\zeta(s)$ extends to an analytic function in the region $\mathrm{Re}(s) > 0$.

(c) Prove the following Taylor expansion for ζ near 1:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n\geq 1} a_n(s-1)^n, \tag{2}$$

for a sequence of complex numbers $\{a_1, a_2, ...\}$. Here γ as defined in (1) (see Question 4) is the Euler-Mascheroni constant.

(d) Using (2) or otherwise, show that the function $-\frac{\zeta'(s)}{\zeta(s)} - \zeta(s)$ is analytic at $s = 1$ and that its Taylor expansion at $s = 1$ is of the form $-2\gamma + \sum_{n\geq 1} c_n(s-1)^n$, for some complex numbers $c_1, c_2, ...$

Q8 Throughout this question, let $q \in \mathbb{N}$ be a natural number.

(a) Define what it means for a function $\chi : \mathbb{Z} \to \mathbb{C}$ to be a Dirichlet character modulo q.

(b) State the orthogonality relations for Dirichlet characters modulo q.

(c) Prove that, for any integers a and n,

$$\sum_{\chi \bmod q} \chi(n)\overline{\chi(a)} = \begin{cases} \phi(q) & \text{if } n \equiv a \bmod q, \text{ and } \gcd(a,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here the sum is over all Dirichlet characters modulo q and $\phi(q)$ denotes the Euler totient function.

(d) Let χ be a real-valued character modulo q. Let f be an arithmetic function defined by

$$f(n) = \sum_{d|n} \chi(d).$$

Prove that

(i) $f(1) = 1$ and $f(n) \geq 0$ for $n \geq 2$.

(ii) $f(n) \geq 1$ if n is a square.