Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4161-WE01

Title: Algebraic Topology IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 For each of the spaces

(a) the sphere S^2,
(b) complex projective space $\mathbb{C}\mathrm{P}^2$, and
(c) real projective space $\mathbb{R}\mathrm{P}^2$

do the following:

- Describe a finite cell complex that is homeomorphic to the given space.
  It suffices to specify how many k-cells are needed for each non-negative integer k. You do not need to specify the attaching maps.
- Determine the Euler characteristic of the space.
- State the corresponding cellular chain complex.
  Be careful to specify all chain groups and all differentials.
- Compute the homology of this chain complex.

You do not need to justify any of your answers in this question.

Q2 Let $f\colon X \to Y$ be a continuous map between two topological spaces and

$$H_n(f)\colon H_n(X) \to H_n(Y)$$

its induced homomorphism. Consider the following statements:

(a) $H_n(f)$ is injective for all n, if f is injective.
(b) $H_n(f)$ is surjective for all n, if f is surjective.
(c) $H_n(f)$ is an isomorphism for all n, if f is a homeomorphism.
(d) Let $X = Y = S^n$ and suppose $H_n(f)$ is injective. Then f is surjective.

In each case, decide if the statement is true or false. If it is true, prove it. If it is false, find a counterexample and prove that is indeed a counterexample.

In this question, you may use your knowledge of the homology groups of any topological space without justification.

Q3 Let $T^4 = S^1 \times S^1 \times S^1 \times S^1$ be the four-dimensional torus. Let $p_i \colon T^4 \to S^1$ be the projection onto the i-th circle S^1, and let a be a generator of $H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. Let $a_i := p_i^*(a) \in H^1(T^4;\mathbb{Z})$. You are allowed to use the fact that $\{a_1, a_2, a_3, a_4\}$ forms a basis of $H^1(T^4;\mathbb{Z})$ without proving it. You are also allowed to use the fact that $H_1(T^4)$ has no torsion, without proving it. You are furthermore allowed to use the fact that the Euler-characteristic of T^4 is 0.

(a) What is the rank of the free abelian group $H^3(T^4;\mathbb{Z})$? Justify your answer!

(b) Using the Euler characteristic, deduce that the rank of $H^2(T^4;\mathbb{Z})$ is 6. Explain why $H^2(T^4;\mathbb{Z})$ has no torsion.

(c) Without justifying your result, provide a basis of $H^2(T^4;\mathbb{Z})$ in terms of cup-products of the elements a_1, a_2, a_3, a_4.

(d) Without justifying your result, provide a basis of $H^3(T^4;\mathbb{Z})$ in terms of cup-products of the elements a_1, a_2, a_3, a_4.

(e) Without justifying your result, provide a basis of $H^4(T^4;\mathbb{Z})$ in terms of cup-products of the elements a_1, a_2, a_3, a_4.

In this question, you may use that T^4 is a closed, connected, orientable manifold, and results about manifolds you have learnt in class.

Q4 Decide for each of the following statements about cup-products whether they hold or not. Give a proof of your assertions. Here T^2 denotes the 2-dimensional torus $S^1 \times S^1$, and $[M] \in H_n(M;\mathbb{Z})$ denotes the fundamental class of a closed, connected, oriented manifold M of dimension n. You are allowed to quote computations performed in class.

(a) For all $a \in H^1(T^2;\mathbb{Z})$ one has

$$\langle\, a \smile a\,, [T^2]\,\rangle = 0.$$

(b) For all $a \in H^2(S^2 \times S^2;\mathbb{Z})$ one has

$$\langle\, a \smile a\,, [S^2 \times S^2]\,\rangle = 0.$$

(c) For all $a \in H^1(\mathbb{R}\mathrm{P}^2;\mathbb{Z})$ one has

$$a \smile a = 0.$$

SECTION B

Q5 For $n > m \geq 0$, consider S^m as the closed subspace of S^n given by

$$\left\{ (x_0, x_1, \ldots, x_m, 0, \ldots, 0) \in \mathbb{R}^{n+1} \;\middle|\; 1 = \sum_{i=0}^{m} x_i^2 \right\}.$$

Likewise, for $\ell = n - m - 1$, consider S^ℓ as the closed subspace of S^n given by

$$\left\{ (0, \ldots, 0, x_{m+1}, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; 1 = \sum_{i=m+1}^{n} x_i^2 \right\}.$$

(a) Show that $(S^n \smallsetminus S^m, S^\ell)$ is a good pair. In other words, show that $S^n \smallsetminus S^m$ deformation retracts onto to S^ℓ.

(b) Compute the reduced homology of S^n/S^ℓ for all integers ℓ with $0 \leq \ell < n$.

In this question, you may use your knowledge of the (reduced) homology groups of spheres of any dimension without justification.

Q6 (a) Consider the following commutative diagram in which the two rows are exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, "\psi_{n+1}"] & A_n \arrow[r, "\alpha_n"] \arrow[d, "1_{A_n}"] & B_n \arrow[r, "\beta_n"] \arrow[d, "\varphi_n"] & C_n \arrow[r, "\gamma_n"] \arrow[d, "\psi_n"] & A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "1_{A_{n-1}}"] & B_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \\
\cdots \arrow[r] & E_{n+1} \arrow[r, "\varepsilon_{n+1}"] & A_n \arrow[r, "\eta_n"] & D_n \arrow[r, "\delta_n"] & E_n \arrow[r, "\varepsilon_n"] & A_{n-1} \arrow[r, "\eta_{n-1}"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Find an exact sequence of the form

$$\cdots \xrightarrow{g_{n+1}} E_{n+1} \xrightarrow{h_{n+1}} B_n \xrightarrow{f_n} C_n \oplus D_n \xrightarrow{g_n} E_n \xrightarrow{h_n} B_{n-1} \xrightarrow{f_{n-1}} \cdots$$

Verify exactness at B_n and $C_n \oplus D_n$ (but not necessarily at E_n).

(b) Suppose X, Y, and Z are three topological spaces satisfying $X \subseteq Y \subseteq Z$. Assuming exactness of the sequence found in part (a), show that there exists a long exact sequence of the form

$$\cdots \to H_{n+1}(Z, X) \to H_n(Y) \to H_n(Y, X) \oplus H_n(Z) \to H_n(Z, X) \to H_{n-1}(Y) \to \cdots$$

(c) Deduce that if the inclusion $X \hookrightarrow Z$ is a homotopy equivalence, then

$$H_n(Y) \cong H_n(Y, X) \oplus H_n(X)$$

for all $n \in \mathbb{Z}$.

Q7 Consider the sequence of abelian groups

$$0 \to C_2 := \mathbb{Z}/12 \xrightarrow{\partial_2} C_1 := \mathbb{Z}/12 \xrightarrow{\partial_1} C_0 := \mathbb{Z}/12 \to 0,$$

where $\partial_2 := \cdot 3\colon \mathbb{Z}/12 \to \mathbb{Z}/12$ denotes the map induced from multiplication with 3 on $\mathbb{Z}$, and likewise $\partial_1 := \cdot 8\colon \mathbb{Z}/12 \to \mathbb{Z}/12$ denotes the map induced from multiplication with 8 on $\mathbb{Z}$.

(a) Show that this sequence is a chain complex of abelian groups (C_*, ∂_*).

(b) Determine its homology groups $H_*(C_*)$.

(c) Determine the cohomology groups $H^*(C^*; \mathbb{Z}/6)$ of the dual complex with coefficients in $\mathbb{Z}/6$.

(d) State the Universal Coefficient Theorem.

(e) Compute the groups $\mathrm{Hom}(H_i(C_*), \mathbb{Z}/6)$ for $i = 0, 1, 2$.

(f) Compute the groups $\mathrm{Ext}^1(H_i(C_*), \mathbb{Z}/6)$ for $i = 0, 1, 2$.

(g) Is it the case that we have isomorphisms

$$H^i(C_*, \mathbb{Z}/6) \cong \mathrm{Hom}(H_i(C_*), \mathbb{Z}/6) \oplus \mathrm{Ext}^1(H_{i-1}(C_*), \mathbb{Z}/6)$$

for $i = 0, 1, 2$? Does this result contradict the Universal Coefficient Theorem?

Q8 (a) State the Poincaré duality theorem.

(b) Prove that the 4-dimensional real projective space $\mathbb{R}\mathrm{P}^4$ is not an orientable manifold.

(c) Is there a closed, connected, orientable 3-dimensional manifold Y with homology groups $H_0(Y) \cong \mathbb{Z}$, $H_1(Y) \cong \mathbb{Z}/2$, $H_2(Y) \cong 0$, and $H_3(Y) \cong \mathbb{Z}$?

(d) Is there a closed, connected, orientable 3-dimensional manifold Y with homology groups $H_0(Y) \cong \mathbb{Z}$, $H_1(Y) \cong \mathbb{Z}/2$, $H_2(Y) \cong \mathbb{Z}/2$, and $H_3(Y) \cong \mathbb{Z}$?

(e) Suppose X is a closed, connected, orientable 4-dimensional manifold with Euler characteristic $\chi(X) = 4$ and $H^3(X) = 0$. Determine all homology groups $H_*(X)$. Give two examples of distinct 4-manifolds with these homology groups!

(f) Let M and N be closed, connected manifolds of dimension m and n, respectively. If one of M or N is non-orientable, or if both are non-orientable, can the product manifold $M \times N$ be orientable?