EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH41620-WE01

Title:
Number Theory V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 30%, Section B is worth 60%, and Section C is worth 10%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Let $\alpha \in \mathbb{C}$ be an algebraic number.

(a) Give the definition of the degree $\deg \alpha$ of α over $\mathbb{Q}$.

(b) Show that if $K/\mathbb{Q}$ is a finite field extension and $\alpha \in K$, then $\deg \alpha \leq [K : \mathbb{Q}]$. (Explicitly mention any result from the lectures that you use.)

(c) Show that if $\beta \in \mathbb{C}$ is another algebraic number, then $\deg(\alpha+\beta) \leq \deg(\alpha)\deg(\beta)$.

Q2 Let $K = \mathbb{Q}(\theta)$ where θ is a root of the irreducible polynomial $x^3 - x + 1$.

(a) Find the matrix T_{θ^2} of multiplication by θ^2 with respect to the basis $\{1, \theta, \theta^2\}$.

(b) Hence, or otherwise, show that the discriminant of $\mathbb{Z}[\theta]$ is -23.

You may use the formula $\Delta_K(\mathbb{Z}[\theta]) = (-1)^{\binom{n}{2}} N_{K/\mathbb{Q}}(p'(\theta))$.

(c) Deduce that $\mathcal{O}_K = \mathbb{Z}[\theta]$.

Q3 Let $K = \mathbb{Q}(\theta)$ where θ is a root of the irreducible polynomial $x^3 - x + 1$, and assume that $\mathcal{O}_K = \mathbb{Z}[\theta]$.

(a) Factorise the ideals (3) and (5) as products of prime ideals of $\mathcal{O}_K$.

(b) Find the norm of each prime ideal occurring in part (a).

(c) Show that the ideal $(7, 2 - \theta)$ of $\mathcal{O}_K$ is principal.

SECTION B

Q4 In this question you may use the facts from the lectures that $\mathbb{Z}[\sqrt{-2}]$ is a UFD and that if α and β are two relatively prime elements of $\mathbb{Z}[\sqrt{-2}]$ such that $\alpha\beta = \gamma^3$, for some $\gamma \in \mathbb{Z}[\sqrt{-2}]$, then $\alpha = \gamma_1^3$ and $\beta = \gamma_2^3$ for some $\gamma_1, \gamma_2 \in \mathbb{Z}[\sqrt{-2}]$.

(a) Let $a \in \mathbb{Z}$. Show that if $\pi \in \mathbb{Z}[\sqrt{-2}]$ is an irreducible element that divides both $a + 2\sqrt{-2}$ and $a - 2\sqrt{-2}$, then

$$\pi = \pm\sqrt{-2}.$$

Conclude that $a + 2\sqrt{-2}$ and $a - 2\sqrt{-2}$ are relatively prime if a is odd.

(b) Find all the solutions $(x, y) \in \mathbb{Z}^2$, if any, to $y^2 + 8 = x^3$ when y is *odd*.

(c) Find all the solutions $(x, y) \in \mathbb{Z}^2$, if any, to $y^2 + 8 = x^3$ when y is *even*.

Q5 (a) Find a factorisation of $-13 + 5\sqrt{-5}$ into irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.

(b) It is known that $\mathbb{Z}[i]$ is a Euclidean domain with respect to the Euclidean function $\phi(a + bi) = a^2 + b^2$. Let $x = 44 + 3i$ and $y = 3 - i$ be elements in $\mathbb{Z}[i]$ and find q and r in $\mathbb{Z}[i]$ such that $x = qy + r$ with either $r = 0$ or $\phi(r) < \phi(y)$.

(c) Let R be a Euclidean domain with Euclidean function ϕ, as defined in the lectures. Show that $x \in R \setminus \{0\}$, is a unit if and only if $\phi(x) = \phi(1)$.

Q6 Let S be the ring $\mathbb{Z}[\sqrt{5}]$. Let J be the ideal $(2, 1+\sqrt{5})$ in S.

(a) Show that J is a maximal ideal of S with $(2) \subset J$.

(b) Show that, if I is an ideal of S with

$$(2) \subsetneq I \subsetneq S,$$

then $I = J$.

(c) Show that $(2) \neq IJ$ for any ideal $I \subset S$ and deduce that the ideal (2) does not have a factorisation as a product of prime ideals in S.

Q7 Let $K = \mathbb{Q}(\sqrt{-47})$.

(a) Find the factorisations of the ideals (2), (3), and $\left(\dfrac{1+\sqrt{-47}}{2}\right)$ inside $\mathcal{O}_K$.

(b) Find the class group of K.
You may use the Minkowski bound, $B_K = \left(\frac{4}{\pi}\right)^t \frac{n!}{n^n} \sqrt{|\Delta_K|}$.

SECTION C

Q8 (a) Show that for any non-archimedean valuation (also called an absolute value) $|\cdot| : F \to \mathbb{R}$ on a field F and any $\alpha \in \mathbb{R}$, with $\alpha > 0$, the function $|\cdot|^\alpha$ is also a non-archimedean valuation.

(b) Compute $|7/54|_3$ and $|123/48|_3$, where $|\cdot|_3$ is the 3-adic valuation on $\mathbb{Q}$ such that $|3|_3 = 1/3$.

(c) Determine the 5-adic expansion of $2/3$.