

EXAMINATION PAPER

Examination Session: May/June Year: 2024

Exam Code:

MATH4171-WE01

Title:

Riemannian Geometry IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:

SECTION A

- Q1 Prove or disprove the following assertions:
 - **1.1** The subset of \mathbb{R}^3 given by

$$M = \{ (x, y, z) \in \mathbb{R}^3 \mid 4x^2 + 6y^2 + 3z^2 = 2 \}$$

- is a smooth 2-dimensional submanifold of \mathbb{R}^3 .
- **1.2** The subset of \mathbb{R}^3 given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid (x-1)^2 + y^2 + z^2 = 1\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid (x+1)^2 + y^2 + z^2 = 1\}$$

is a smooth submanifold of \mathbb{R}^3 .

- **Q2** 2.1 State the definition of the tangent bundle TM of a differentiable manifold M.
 - **2.2** Let M be an n-dimensional differentiable manifold. Show that TM is 2n-dimensional. You may assume that TM is a differentiable manifold.
 - 2.3 Let

$$\begin{split} X(x,y,z) &= x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}\\ Y(x,y,z) &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}. \end{split}$$

be two differentiable vector fields on \mathbb{R}^3 equipped with the usual chart given by the identity. Compute the Lie bracket [X, Y].

- Q3 Prove or disprove the following assertions:
 - **3.1** The smooth manifolds $S^2 \times S^2$ and $S^2 \times S^3$ are diffeomorphic.
 - **3.2** There exists a Riemannian metric g on \mathbb{R}^2 such that $(g_{ij}(p))$, the expression in local coordinates of g at some point $p \in \mathbb{R}^2$ for some given coordinate chart around p, is the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- **3.3** The smooth manifold $S^1 \times \mathbb{R}$ admits a complete Riemannian metric with sectional curvature $K \geq 1$.
- **Q4** Let (M, g) be a Riemannian manifold.
 - **4.1** Prove or disprove the following assertion: The Levi-Civita connection is a $\binom{2}{1}$ tensor field on M.
 - **4.2** State the Jacobi equation for a differentiable vector field V along a geodesic $c: [a, b] \to M$.
 - **4.3** Let $J: [a, b] \to TM$ be a Jacobi field along a geodesic $c: [a, b] \to M$ and let $f(t) = ||J(t)||^2$. Show that if M has non-positive sectional curvature, then $f''(t) \ge 0$.



SECTION B

- **Q5** 5.1 Let (M, g) be an *n*-dimensional complete Riemannian manifold, fix $p \in M$ and suppose that $\exp_p: T_pM \to M$ is a diffeomorphism. Let $\varphi: T_pM \to T_pM$ be an orthogonal transformation and define $f: M \to M$ by $f = \exp_p \circ \varphi \circ \exp_p^{-1}$. Show that Df_p is a linear isomorphism of T_pM given by $Df_p = \psi \circ \varphi \circ \psi^{-1}$, where ψ is a linear isomorphism of T_pM .
 - **5.2** Let (M, g) be a Riemannian manifold and let $f: M \to \mathbb{R}$ be a smooth function. The *gradient* of f, denoted by grad f, is the smooth vector field on M characterised by

$$Y(f) = \langle \operatorname{grad} f, Y \rangle$$

for all $Y \in \mathcal{X}(M)$. Show that, if X and Y are smooth vector fields on M, then

$$\langle \nabla_X \operatorname{grad} f, Y \rangle = \langle \nabla_Y \operatorname{grad} f, X \rangle.$$

You may use in your computations the fact that the Levi–Civita connection is torsion-free and compatible with the Riemannian metric, as well as the properties of the Lie bracket.

- **5.3** Let (M, g) be a Riemannian manifold and let $f: M \to \mathbb{R}$ be a smooth function. Consider the smooth vector field X on M given by $X(p) = \operatorname{grad} f(p)$ for all $p \in M$. An *integral curve* of X through $p_o \in M$ is a smooth curve $\gamma: (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p_o$ and $\gamma'(t) = X(\gamma(t))$ for all $t \in (-\varepsilon, \varepsilon)$. Use the identity in question **5.2** to show that, if $||\operatorname{grad} f(p)|| = c > 0$ for all $p \in M$, then the integral curves of X are geodesics.
- **Q6** Consider the smooth manifold $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ with the global chart given by the identity map. Let g be the Riemannian metric on M given by $g_{11}(x, y) = 1$, $g_{12}(x, y) = g_{21}(x, y) = 0$, and $g_{22}(x, y) = e^{2x}$.
 - **6.1** Compute the sectional curvature of M.
 - **6.2** Let a, b, c > 0. Show that the differentiable curve $\gamma: (0, \infty) \to M$ given by $\gamma(t) = (at + b, c)$ is a geodesic in M.
- **Q7** Fix a positive integer n and let $M(2n, \mathbb{R})$ denote the set of $2n \times 2n$ real matrices. Let Id_n be the $n \times n$ identity matrix and define

$$\operatorname{Sp}(2n) = \{ A \in M(2n, \mathbb{R}) \mid A^T \Omega A = \Omega \},\$$

where

$$\Omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}.$$

The elements of Sp(2n) are called $2n \times 2n$ symplectic matrices.

- **7.1** Show that Sp(2n) is a group under matrix multiplication.
- **7.2** Show that Sp(2) is a Lie group and determine its dimension.
- **7.3** Express T_{Id_2} Sp(2) as a subset of $M(2, \mathbb{R})$.





- **Q8 8.1** State the definition of a local isometry between two Riemannian manifolds (M,g) and (N,h).
 - 8.2 Let \widetilde{M} and M be smooth connected manifolds. A map $\pi \colon \widetilde{M} \to M$ is a covering map if π is smooth and surjective, and each point in M has a neighbourhood U such that π maps each component of $\pi^{-1}(U)$ diffeomorphically onto U. Suppose that M has a Riemannian metric g. Show that \widetilde{M} has a Riemannian metric \widetilde{g} such that the covering map $\pi \colon \widetilde{M} \to M$ is a local isometry.
 - **8.3** Let M be a differentiable manifold and let ∇ be an affine connection on M. Show that the map $T: \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

defines a $\binom{2}{1}$ tensor field on M. You may use without proof the properties of an affine connection and of the Lie bracket.