Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4241-WE01

Title:
Representation Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Let $(\pi, \mathbb{C}^6)$ denote the permutation representation of S_6 on $\mathbb{C}^6$.

(a) Find a three-dimensional subspace U of $\mathbb{C}^6$ such that $(\mathrm{Res}^{S_6}_{S_4}\pi, U)$ is isomorphic to the trivial representation $(\mathrm{Id}, \mathbb{C}^3)$ of S_4.

(b) Compute $\|\mathrm{Res}^{S_6}_{S_4}\chi_\pi\|_{S_4}$.

Q2 Let G be the group of order twenty with five conjugacy classes.

size:	1	4	5	5	5
class	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$

(a) Compute the entries of the $\mathcal{C}_1$-column of the character table of G.

(b) Letting $H = \mathcal{C}_1 \cup \mathcal{C}_2$ be the normal subgroup of G such that $\widetilde{G} = G/H \cong C_4$, use lifts of the irreducible representations of $\widetilde{G}$ to G to find four rows of the character table of G.

(c) Complete the character table of G.

Q3 (a) Describe the following Lie algebras as subalgebras of some space of matrices and calculate their dimension as a real vector space: $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{so}(n)$, and $\mathfrak{su}(n)$.

(b) Give a basis for each of $\mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$.

Q4 (a) Describe, without proof, all the finite dimensional irreducible representations up to isomorphism

(i) of $U(1)$,

(ii) of $SU(2)$.

(b) Compute $\exp(I + tX)$ for the matrix $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $t \in \mathbb{R}$, where I denotes the (3×3)-identity matrix.

SECTION B

Q5 For $n \geq 2$, let $(\pi, \mathcal{M}^{(n-1,1)})$ denote the permutation representation of S_n on the space of polytabloids of shape $(n-1, 1)$.

(a) Show that

$$(\pi, \mathcal{M}^{(n-1,1)}) \cong (\pi, \mathcal{S}^{(n)}) \oplus (\pi, \mathcal{S}^{(n-1,1)}),$$

where $\mathcal{S}^{(n)}$ is the Specht module of shape (n) and $\mathcal{S}^{(n-1,1)}$ is the Specht module of shape $(n-1, 1)$.

(b) Show that

$$\|\mathrm{Ind}^{S_{n+1}}_{S_n}\chi_{(\pi,\mathcal{M}^{(n-1,1)})}\|_{S_{n+1}} = \begin{cases} \sqrt{6} & \text{if } n = 2, \\ \sqrt{7} & \text{otherwise.} \end{cases}$$

Q6 Given a prime number p, let G be a group of order p^3 such that $|Z(G)| = p$. Here $Z(G)$ denotes the centre of G.

(a) State the character formula and use it to show that for each representation π of $Z(G)$,

$$\mathrm{Ind}_{Z(G)}^{G}\chi_\pi(g) = \begin{cases} p^2 \cdot \chi_\pi(g) & \text{if } g \in Z(G), \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that

$$\{\mathrm{Ind}_{Z(G)}^{G}\chi_\pi \,:\, \pi \in \mathrm{Irr}(Z(G)),\ \pi \ncong \mathrm{triv}\}$$

is an orthonormal set in the space of class functions on G.

(c) Use part (b) and lifts of representations of $G/Z(G)$ to show that G has at least $p^2 + p - 1$ isomorphism classes of irreducible representations. *Hint: recall that any group of order p^2 is Abelian.*

(d) Deduce from (b), (c), and the fact that G has $p^2 + p - 1$ conjugacy classes that G has $p - 1$ irreducible representations of degree p.

Q7 Consider a finite dimensional representation (ρ, V) of $\mathfrak{sl}_2(\mathbb{C})$.

(a) Write down the standard basis elements X, Y, and H of $\mathfrak{sl}_2(\mathbb{C})$ as matrices, and express the Lie brackets $[X, Y]$, $[H, X]$, and $[H, Y]$ in this basis.

(b) Show that

$$Z := 2\rho(X)\rho(Y) + 2\rho(Y)\rho(X) + \rho(H)\rho(H)$$

commutes with the action of $\mathfrak{sl}_2(\mathbb{C})$.

(c) Now suppose V is irreducible. Show that the element Z defined above then acts as a scalar. Determine this scalar for the irreducible representation of highest weight n.

(d) Let $V = \mathbb{C}[x]$ be the (infinite-dimensional) $\mathbb{C}$-vector space of complex polynomials in one variable. For a certain representation (ρ, V) we are given that, for any $f \in V$

$$(\rho(X)f)(x) = f'(x)$$

and

$$(\rho(Y)f)(x) = xf(x)\,,$$

where $f'(x)$ denotes the derivative of $f(x)$ with respect to x.
What is Z for this representation?

Q8 For given $n > 0$, consider $V = V^{(n)} = \mathrm{Sym}^n(\mathbb{C}^2)$, the irreducible representation of highest weight n of $\mathfrak{sl}_2(\mathbb{C})$. Let $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, acting as a lowering operator, let v_n be a highest weight vector and set $v_{n-2k} = Y^k(v_n)$ with $k = 0, \ldots, n$.

(a) State the Clebsch-Gordan formula expressing $\mathrm{Sym}^a(\mathbb{C}^2) \otimes \mathrm{Sym}^b(\mathbb{C}^2)$ for any $a \geq b \geq 0$.

(b) Decompose the representations defined by $V \otimes V \otimes V$ for $V = V^{(1)} = \mathbb{C}^2$ and by $\mathrm{Sym}^3(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2)$, and give an explicit weight basis in each case.

(c) Exhibit a highest weight vector for each of the irreducible constituents in the latter representation $\mathrm{Sym}^3(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2)$, using the v_{n-2k} $(k = 0, \ldots, n)$.

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