EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4261-WE01

Title: Stochastic Analysis IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

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SECTION A

Q1 Fix real constants $\mu < 0$ and $\sigma^2 > 0$. Let $\zeta_1, \zeta_2, \dots$ be independent, identically distributed $\mathcal{N}(\mu, \sigma^2)$ random variables.

Let $S_0 := 0$ and $S_n := \sum_{i=1}^n \zeta_i$ for $n \in \mathbb{N}$. Also consider

$$S^\star := \sup_{n \in \mathbb{Z}_+} S_n.$$

(a) Show that there is a unique $\lambda > 0$ (which you should identify as a function of μ and σ^2) such that $M_n := \mathrm{e}^{\lambda S_n}$ defines a discrete-time martingale with respect to the natural filtration associated with S_n.

(b) For $y \in \mathbb{Z}_+$, consider the stopping time $\tau_y := \inf\{n \in \mathbb{Z}_+ : S_n > y\}$. Using the optional stopping theorem and the martingale identified in part (a), prove that

$$1 \geq \mathrm{e}^{\lambda y}\, \mathbb{P}(\tau_y \leq n), \text{ for all } n \in \mathbb{Z}_+,$$

where λ is the constant from part (a).

(c) Deduce that

$$\mathbb{P}\left(S^\star > y\right) \leq \mathrm{e}^{-\lambda y}, \text{ for all } y \in \mathbb{Z}_+.$$

Q2 Let ν, μ, and ρ be finite measures on a measurable space $(\Omega, \mathcal{F})$.

(a) Explain what it means to say that ν is absolutely continuous with respect to μ, written $\nu \ll \mu$.

(b) Show that if $\nu \ll \mu$ and $\mu \ll \rho$, then $\nu \ll \rho$.

One version of the Radon–Nikodym theorem states that if $\nu \ll \mu$, then there exists a (Radon–Nikodym) derivative $\mathrm{d}\nu / \mathrm{d}\mu : \Omega \to \mathbb{R}_+$ (measurable) such that

$$\int f \,\mathrm{d}\nu = \int f \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\mu \tag{1}$$

for all measurable functions $f : \Omega \to \mathbb{R}_+$.

(c) Prove that the Radon–Nikodym derivative appearing in (1) is essentially unique, in a sense that you should state precisely.

(d) Deduce that if $\nu \ll \mu$ and $\mu \ll \rho$, then the corresponding derivatives satisfy

$$\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}\rho},$$

up to sets of measure zero, in a sense that you should state precisely.

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Q3 Let $W_t, t \in \mathbb{R}_+$, be a Brownian motion on $\mathbb{R}$ and define the martingale $X_t, t \in \mathbb{R}_+$, by

$$X_t = \int_0^t \mathrm{e}^{-s}\,\mathrm{d}W_s.$$

(a) State the definition of uniform integrability and prove that X given above is a uniformly integrable martingale by verifying the definition.

(b) Use the down-crossing inequality to prove that $\lim_{t\to+\infty} X_t = X_\infty$ exists almost surely.

(c) State a theorem that guarantees the convergence in Part (b) also holds in L^1 and use the L^1 convergence to prove that

$$X_t = \mathbb{E}[X_\infty | \mathcal{F}_t], \quad \text{a.s.}$$

where $\mathcal{F}_t = \sigma(W_s, 0 \le s \le t)$.

Q4 (a) Using Itô's formula, prove that if $M_t, t \in \mathbb{R}_+$, is a continuous local martingale, and $\lambda \in \mathbb{R}$ is a constant, then the process

$$\mathcal{E}^\lambda(M)_t = \exp\left(\lambda M_t - \frac{\lambda^2}{2}\langle M, M\rangle_t\right), \ t \in \mathbb{R}_+,$$

is a local martingale.

(b) If $M_0 = 0$, prove that $\mathcal{E}^\lambda(M)$ is a martingale if and only if $\mathbb{E}[\mathcal{E}^\lambda(M)_t] = 1$ for every $t \in \mathbb{R}_+$.

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SECTION B

Q5 Let $W = (W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion. For $t \in \mathbb{R}_+$, define

$$U_t := 2W_{1+(t/2)} - W_1 - W_{1+t}.$$

In your answers to the following parts, you may use, without proof, results from lectures on multivariate normal distributions and/or Brownian motion, but you should state them clearly.

(a) Find the distribution of U_t for $t \in \mathbb{R}_+$.

(b) For $s \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$, show that (W_{1+s}, U_t) has a bivariate normal distribution, and compute $\mathbb{E}(W_{1+s}U_t)$.

(c) Deduce that, for every $t \in \mathbb{R}_+$, U_t is independent of W_1, and that, for every $t \in \mathbb{R}_+$, U_t is independent of W_{1+t}.

(d) Compute $\mathbb{E}(U_sU_t)$ for $s, t \in \mathbb{R}_+$.

(e) Is $U = (U_t)_{t\in\mathbb{R}_+}$ a Brownian motion? Justify your answer.

Q6 Let $\xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed random variables with $\mathbb{P}(\xi_i = +1) = \mathbb{P}(\xi_i = -1) = 1/2$.

Define $S_0 := 0$ and $S_n := \sum_{i=1}^{n} \xi_i$ for $n \in \mathbb{N}$.

For a positive integer a, define $\tau_a := \inf\{n \in \mathbb{Z}_+ : S_n = a\}$.

Fix $\theta \in \mathbb{R}$ and define

$$X_n := \frac{\mathrm{e}^{\theta S_n}}{(\cosh\theta)^n}, \text{ for all } n \in \mathbb{Z}_+.$$

(a) Show that, for any $\theta \in \mathbb{R}$, $X = (X_n)_{n\in\mathbb{Z}_+}$ is a martingale with respect to the natural filtration associated with S_n.

(b) For $\theta > 0$, prove that, almost surely,

$$\lim_{n\to\infty} X_{n\wedge\tau_a} = \frac{\mathrm{e}^{\theta a}}{(\cosh\theta)^{\tau_a}} \mathbb{1}\{\tau_a < \infty\}.$$

(c) Give a justification of the fact that

$$\lim_{\theta\to 0} \left(\frac{\mathrm{e}^{\theta a}}{(\cosh\theta)^{\tau_a}}\right) \mathbb{1}\{\tau_a < \infty\} = \mathbb{1}\{\tau_a < \infty\}.$$

(d) Prove that $\mathbb{P}(\tau_a < \infty) = 1$ and, for every $\theta \geq 0$,

$$\mathbb{E}\left[(\operatorname{sech}\theta)^{\tau_a}\right] = \mathrm{e}^{-\theta a},$$

where $\operatorname{sech} y := 1/\cosh y$.

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Q7 In your answers to the following parts, you may use, without proof, the result that any bounded and continuous martingale M is of finite quadratic variation and $\langle M, M\rangle$ is the unique continuous, adapted and increasing process such that $M^2 - \langle M, M\rangle$ is a martingale.

Now let $M_t, t \in \mathbb{R}_+$, and $N_t, t \in \mathbb{R}_+$, be two continuous real-valued local martingales.

(a) Prove that there is a unique continuous increasing adapted process $\langle M, M\rangle$, vanishing at zero, such that $M^2 - \langle M, M\rangle$ is a continuous local martingale.

(b) Let $\{\Delta_n\}$ be a sequence of subdivisions of $\mathbb{R}_+$, each with only a finite number of points in $[0, t]$ for each $t \geq 0$. Prove that as $|\Delta_n| \to 0$, for each $t \geq 0$

$$\sup_{s \leq t} |T_s^{\Delta_n}(M) - \langle M, M\rangle_s| \to 0,$$

in probability. Here

$$T_t^{\Delta_n}(M) = \sum_{i=0}^{k-1} (M_{t_{i+1}} - M_{t_i})^2 + (M_t - M_{t_k})^2,$$

where $\Delta_n = \{t_0 = 0 < t_1 < t_2 < \cdots\}$ and k is the unique integer such that $t_k \leq t < t_{k+1}$.

(c) Prove that there is a unique continuous process $\langle M, N\rangle$ of finite variation, vanishing at zero, such that $MN - \langle M, N\rangle$ is a local martingale.

Q8 Let $W_t, t \in \mathbb{R}_+$, be a Brownian motion on $\mathbb{R}$ and define for any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$,

$$P_t f(x) = \mathbb{E}[f(x + W_t)], \quad t \in \mathbb{R}_+, x \in \mathbb{R}.$$

(a) Prove, using the Markov property, that for any $t, s \geq 0$, and f being bounded and measurable,

$$P_t \circ P_s f = P_{t+s} f.$$

(b) Let A be the infinitesimal generator of $X_t^x = x + W_t$, $t \in \mathbb{R}_+$. Prove that $Af = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} f$ when $f \in C^2$ is bounded and has bounded continuous derivatives up to second order.

(c) Prove that for any $t > 0$, for f being bounded and measurable, $P_t f$ is in the domain of A and

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t f = A P_t f.$$

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