



EXAMINATION PAPER

Examination Session: May/June	Year: 2024	Exam Code: MATH4271-WE01
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Title: Superstrings IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.</p> <p>Students must use the mathematics specific answer book.</p>
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Revision:	
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SECTION A

Q1 Consider the following classical trajectory of an open string:

$$X^0 = \omega\tau, \quad X^1 = \cos(\omega\tau) \cos(\omega\sigma), \quad X^2 = \sin(a\tau) \cos(b\sigma),$$

and assume conformal gauge.

1.1 Determine the values of a and b so that the open string solves the equations of motion $\left(\ddot{X} - X''\right)^\mu = 0$ and the Virasoro constraints $\left(\dot{X} \pm X'\right)^2 = 0$.

1.2 Recalling that the endpoints of the string are at $\sigma = 0$ and $\sigma = \pi$, what constraint does ω need to satisfy in order for the string to have Neuman boundary conditions?

Q2 For a closed bosonic string we may write $X^\mu(z, \bar{z}) = X^\mu(\bar{z}) + X^\mu(z)$ where $X^\mu(z)$ and $X^\mu(\bar{z})$ have the following operator product expansions:

$$X^\mu(z)X^\nu(w) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-w) + \dots \quad (1)$$

$$X^\mu(\bar{z})X^\nu(\bar{w}) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(\bar{z}-\bar{w}) + \dots \quad (2)$$

$$X^\mu(z)X^\nu(\bar{w}) = \dots, \quad (3)$$

where ... denote non-singular terms in the expansion.

2.1 Use (1)-(3) to derive the following operator product expansions:

$$T(z)X^\mu(w, \bar{w}) = \frac{1}{z-w}\partial X^\mu(w, \bar{w}) + \dots, \quad \tilde{T}(\bar{z})X^\mu(w, \bar{w}) = \frac{1}{\bar{z}-\bar{w}}\bar{\partial}X^\mu(w, \bar{w}) + \dots \quad (4)$$

where

$$T(z) = -\frac{1}{\alpha'} : \partial X(z) \cdot \partial X(z) :, \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X(\bar{z}) \cdot \bar{\partial} X(\bar{z}) :.$$

2.2 Use (4) to compute the singular part of the operator product expansion of T and \tilde{T} with the following operators:

$$\partial X^\mu(w, \bar{w}), \quad \bar{\partial} X^\mu(w, \bar{w}), \quad \partial^2 X^\mu(w, \bar{w}).$$

Which of these operators are conformal primaries? What are their conformal weights?

Q3 Consider the Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I},$$

with \mathbb{I} the identity operator and $\eta^{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$ the standard D -dimensional Minkowski metric. Show that in any unitary representation of the Clifford algebra, i.e. any representation such that $(\Gamma^\mu)^\dagger = (\Gamma^\mu)^{-1}$, we must have

$$(\Gamma^0)^\dagger = -\Gamma^0, \quad (\Gamma^i)^\dagger = \Gamma^i, \quad i = 1, \dots, D-1.$$

Then deduce the identity

$$(\Gamma^\mu)^\dagger = \Gamma^0 \Gamma^\mu \Gamma^0.$$

Q4 Consider a Grassmann operator \hat{Q} which is the generator of a fermionic symmetry. On any field operator $\hat{\Phi}$ (bosonic or fermionic) the infinitesimal symmetry transformation generated by \hat{Q} is given by $\delta\hat{\Phi} = i[\epsilon\hat{Q}, \hat{\Phi}]$ with ϵ an infinitesimal Grassmann parameter.

Show that

$$\delta\hat{\Phi} = \begin{cases} i\epsilon [\hat{Q}, \hat{\Phi}], & \text{for } \hat{\Phi} \text{ bosonic,} \\ i\epsilon \{\hat{Q}, \hat{\Phi}\}, & \text{for } \hat{\Phi} \text{ fermionic.} \end{cases}$$

SECTION B

Q5 Recall the Nambu-Goto action

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu})},$$

where $G_{\mu\nu}$ is the background metric. Consider a closed string in a background metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2.$$

5.1 Let $t = \tau$ and $\theta = \sigma$, and $r = r(\tau)$. Show that the action is given by

$$S = -\frac{1}{\alpha'} \int dt r \sqrt{1 - \dot{r}^2}.$$

5.2 Let $\mathcal{L} = -\frac{1}{\alpha'} r \sqrt{1 - \dot{r}^2}$. Show that the Hamiltonian $\mathcal{H} = \Pi_r \dot{r} - \mathcal{L}$ can be written as

$$\mathcal{H} = \sqrt{\Pi_r^2 + V(r)},$$

where $\Pi_r = \partial\mathcal{L}/\partial\dot{r}$ and you must find the potential $V(r)$. Describe qualitatively how the string moves in this potential.

Q6 The closed string scattering amplitude for n tachyons in bosonic string theory takes the form

$$\mathcal{A}_n \propto \int \prod_{i=1}^n d^2 z_i \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l},$$

where the external momenta satisfy $\sum_{i=1}^n p_i^\mu = 0$ and $p_i^2 = 4/\alpha'$. Now consider an $SL(2, \mathbb{C})$ transformation

$$z_i \rightarrow \frac{az_i + b}{cz_i + d},$$

where a, b, c, d are complex numbers satisfying $ad - bc = 1$.

6.1 Show that under the above transformation

$$dz_i \rightarrow \frac{dz_i}{(cz_i + d)^2}$$

$$z_i - z_j \rightarrow \frac{z_i - z_j}{(cz_i + d)(cz_j + d)}.$$

6.2 Show that the n -point tachyon amplitude is $SL(2, \mathbb{C})$ invariant.

Q7 Consider the quantum Hamiltonian

$$\hat{H} = \hat{a}^\dagger \hat{a} + \hat{f}^\dagger \hat{f},$$

where \hat{a}^\dagger, \hat{a} are canonical bosonic creation/annihilation operators and \hat{f}^\dagger, \hat{f} are canonical fermionic creation/annihilation operators, i.e. they satisfy

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1, & [\hat{a}, \hat{a}] &= [\hat{a}^\dagger, \hat{a}^\dagger] = 0, \\ \{\hat{f}, \hat{f}^\dagger\} &= 1, & \{\hat{f}, \hat{f}\} &= \{\hat{f}^\dagger, \hat{f}^\dagger\} = 0, \end{aligned}$$

while all the bosonic operators commute with all the fermionic operators:

$$[\hat{a}, \hat{f}] = [\hat{a}^\dagger, \hat{f}] = [\hat{a}, \hat{f}^\dagger] = [\hat{a}^\dagger, \hat{f}^\dagger] = 0.$$

The Fock vacuum $|0\rangle$ is the unique normalised state annihilated simultaneously by \hat{a} and \hat{f} , i.e.

$$\hat{a} |0\rangle = \hat{f} |0\rangle = 0.$$

7.1 Consider the state

$$|n, 0\rangle = (\hat{a}^\dagger)^n |0\rangle,$$

and compute $\hat{H} |n, 0\rangle$.

7.2 Define the fermionic operator $\hat{Q}^\dagger = \hat{a} \hat{f}^\dagger$ and compute $\hat{Q}^\dagger |n, 0\rangle$.

7.3 Finally compute $\hat{H} \hat{Q}^\dagger |n, 0\rangle$. From this calculation which consequence can you deduce for the spectrum of the Hamiltonian?

Q8 The RNS gauge-fixed Polyakov action is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \eta^{ab} (\partial_a X^\mu \partial_b X^\nu + i \bar{\Psi}^\mu \rho_a \partial_b \Psi^\nu) \eta_{\mu\nu},$$

where $X^\mu(\sigma)$ are D worldsheet bosonic fields, while $\Psi^\mu(\sigma)$ are D worldsheet Majorana fermions. The 2-dimensional Dirac matrices are

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

8.1 Compute the equations of motions for $X^\mu(\sigma)$ and $\Psi^\mu(\sigma)$. You may discard boundary terms.

8.2 Show that the supercurrent

$$J_a = \frac{1}{2} \rho^b \rho_a \Psi^\mu \partial_b X_\mu,$$

satisfies the off-shell (i.e. without using the equations of motion) relation

$$\rho^a J_a = 0.$$

8.3 Show that the supercurrent J^a is conserved on-shell, i.e. show that

$$\partial^a J_a = 0.$$