

EXAMINATION PAPER

Examination Session: May/June

2024

Year:

Exam Code:

MATH4277-WE01

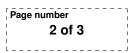
Title:

Discrete & Continuous Probability IV

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.	

Revision:



SECTION A

Q1 Let $(S_k)_{k=0}^{2n}$ be a 2*n*-step trajectory of a simple symmetric random walk starting at the origin (and making jumps ± 1 with probability 1/2) and consider the probabilities

$$u_{2n} = \mathsf{P}(S_{2n} = 0), \quad f_{2n} = \mathsf{P}(S_1 \neq 0, S_2 \neq 0, \dots, S_{2n-1} \neq 0, S_{2n} = 0).$$

(a) Show that $u_{2n} = \binom{2n}{n} 2^{-2n}$ and $f_{2n} = 2 C_{2n-2} 2^{-2n}$, where

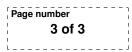
$$C_{2n} = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

are the Catalan numbers.

- (b) Deduce that $f_{2n} = \frac{1}{2n} u_{2n-2} = u_{2n-2} u_{2n}$.
- (c) Describe all real α for which $\sum_{n\geq 1} n^{\alpha} f_{2n} < \infty$.

Q2 Let π be a permutation of the set $\{1, 2, \ldots, n\}$, chosen uniformly at random.

- (a) If $A_m = \{m \text{ is a fixed point of } \pi\}$, find the probability $\mathsf{P}(A_{m_1} \cap A_{m_2} \cap \cdots \cap A_{m_k})$ for distinct $1 \le m_1 < m_2 < \cdots < m_k \le n$.
- (b) Let S_n be the number of fixed points of π . By using inclusion-exclusion or otherwise, find $\mathsf{P}(S_n > 0) \equiv \mathsf{P}(\bigcup_{m=1}^n A_m)$; deduce that $\mathsf{P}(S_n = 0) \to e^{-1}$ as $n \to \infty$.
- (c) Show that, as $n \to \infty$, the distribution of S_n converges to $\mathsf{Poi}(1)$, the Poisson distribution with parameter 1.





SECTION B

- **Q3** For an *n*-sample $\{X_k\}_{k=1}^n$ from the uniform distribution on (0, 1), let $X_{(k)}$ and $\Delta_{(k)}X$ be, respectively, the *k*th order variable and the *k*th gap.
 - (a) For positive a find the limit of $\mathsf{P}(nX_{(1)} > a)$ as $n \to \infty$. What does it tell you about the large-*n* behaviour of $nX_{(1)} \equiv n\Delta_{(1)}X$?
 - (b) By using induction or otherwise, show that

$$\mathsf{P}(\Delta_{(1)}X \ge r_1, \dots, \Delta_{(n+1)}X \ge r_{n+1}) = \left(1 - \sum_{k=1}^{n+1} r_k\right)^n$$

if positive r_k satisfy $\sum_{k=1}^{n+1} r_k \leq 1$. Deduce that all gaps $\Delta_{(k)}X$ have the same distribution. What does it tell you about the typical size of $\Delta_{(k)}X$ for large n?

- (c) Let $\Delta_n^* X = \min_k \Delta_{(k)} X$ be the size of the minimal gap of the *n*-sample under consideration. For positive *a* find the limit of $\mathsf{P}(n^2 \Delta_n^* X > a)$ as $n \to \infty$. What does it tell you about the typical size of the minimal gap $\Delta_n^* X$ for large *n*?
- **Q4** Let $(X_n)_{n\geq 1}$ be independent random variables with common Geom(p) distribution, that is, X_1 takes integer values and satisfies $\mathsf{P}(X_1 > k) = q^k$ with $q = 1 p \in (0, 1)$ and integer $k \geq 0$.
 - (a) Find a constant c such that $\mathsf{P}(\limsup_{n \to \infty} \frac{X_n}{\log n} = c) = 1.$
 - (b) Consider $M_n = \max_{1 \le k \le n} X_k$, the running record value at time *n*. Show that with the same constant *c* as above,

$$\mathsf{P}\Big(\limsup_{n \to \infty} \frac{M_n}{\log n} = c\Big) = 1.$$

(c) Compute the probability $\mathsf{P}(M_n \leq x)$ and use your result to find a constant c such that $\mathsf{P}(\lim_{n \to \infty} \frac{M_n}{\log n} = c) = 1$.

In your answer you should clearly state every result you use.

Hint: You may use without proof the following fact: If $(x_n)_{n\geq 1}$ are real numbers, $m_n \equiv \max_{1\leq k\leq n} x_k$, and a monotone sequence $(b_n)_{n\geq 1}$ increases to infinity as $n \to \infty$, then the sets $\{n \in \mathbb{N} : x_n \geq b_n\}$ and $\{n \in \mathbb{N} : m_n \geq b_n\}$ are both finite or both infinite.