Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4277-WE01

Title: Discrete & Continuous Probability IV

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Let $(S_k)_{k=0}^{2n}$ be a $2n$-step trajectory of a simple symmetric random walk starting at the origin (and making jumps ± 1 with probability $1/2$) and consider the probabilities

$$u_{2n} = \mathsf{P}(S_{2n} = 0)\,, \qquad f_{2n} = \mathsf{P}(S_1 \neq 0, S_2 \neq 0, \dots, S_{2n-1} \neq 0, S_{2n} = 0)\,.$$

(a) Show that $u_{2n} = \binom{2n}{n} 2^{-2n}$ and $f_{2n} = 2\, C_{2n-2}\, 2^{-2n}$, where

$$C_{2n} = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

are the Catalan numbers.

(b) Deduce that $f_{2n} = \frac{1}{2n}\, u_{2n-2} = u_{2n-2} - u_{2n}$.

(c) Describe all real α for which $\sum_{n\geq 1} n^\alpha f_{2n} < \infty$.

Q2 Let π be a permutation of the set $\{1, 2, \dots, n\}$, chosen uniformly at random.

(a) If $A_m = \{m \text{ is a fixed point of } \pi\}$, find the probability $\mathsf{P}\big(A_{m_1} \cap A_{m_2} \cap \dots \cap A_{m_k}\big)$ for distinct $1 \leq m_1 < m_2 < \dots < m_k \leq n$.

(b) Let S_n be the number of fixed points of π. By using inclusion-exclusion or otherwise, find $\mathsf{P}(S_n > 0) \equiv \mathsf{P}\big(\cup_{m=1}^n A_m\big)$; deduce that $\mathsf{P}(S_n = 0) \to e^{-1}$ as $n \to \infty$.

(c) Show that, as $n \to \infty$, the distribution of S_n converges to $\mathsf{Poi}(1)$, the Poisson distribution with parameter 1.

SECTION B

Q3 For an n-sample $\{X_k\}_{k=1}^{n}$ from the uniform distribution on $(0,1)$, let $X_{(k)}$ and $\Delta_{(k)}X$ be, respectively, the kth order variable and the kth gap.

(a) For positive a find the limit of $\mathsf{P}(nX_{(1)} > a)$ as $n \to \infty$. What does it tell you about the large-n behaviour of $nX_{(1)} \equiv n\Delta_{(1)}X$?

(b) By using induction or otherwise, show that

$$\mathsf{P}\big(\Delta_{(1)}X \geq r_1, \ldots, \Delta_{(n+1)}X \geq r_{n+1}\big) = \Big(1 - \sum_{k=1}^{n+1} r_k\Big)^n$$

if positive r_k satisfy $\sum_{k=1}^{n+1} r_k \leq 1$. Deduce that all gaps $\Delta_{(k)}X$ have the same distribution. What does it tell you about the typical size of $\Delta_{(k)}X$ for large n?

(c) Let $\Delta_n^* X = \min_k \Delta_{(k)}X$ be the size of the minimal gap of the n-sample under consideration. For positive a find the limit of $\mathsf{P}(n^2\Delta_n^* X > a)$ as $n \to \infty$. What does it tell you about the typical size of the minimal gap $\Delta_n^* X$ for large n?

Q4 Let $(X_n)_{n\geq 1}$ be independent random variables with common $\mathsf{Geom}(p)$ distribution, that is, X_1 takes integer values and satisfies $\mathsf{P}(X_1 > k) = q^k$ with $q = 1 - p \in (0,1)$ and integer $k \geq 0$.

(a) Find a constant c such that $\mathsf{P}\big(\limsup_{n\to\infty} \frac{X_n}{\log n} = c\big) = 1$.

(b) Consider $M_n = \max_{1\leq k\leq n} X_k$, the running record value at time n. Show that with the same constant c as above,

$$\mathsf{P}\Big(\limsup_{n\to\infty} \frac{M_n}{\log n} = c\Big) = 1\,.$$

(c) Compute the probability $\mathsf{P}(M_n \leq x)$ and use your result to find a constant c such that $\mathsf{P}\big(\lim_{n\to\infty} \frac{M_n}{\log n} = c\big) = 1$.

In your answer you should clearly state every result you use.

Hint: You may use without proof the following fact: *If $(x_n)_{n\geq 1}$ are real numbers, $m_n \equiv \max_{1\leq k\leq n} x_k$, and a monotone sequence $(b_n)_{n\geq 1}$ increases to infinity as $n \to \infty$, then the sets $\{n \in \mathbb{N} : x_n \geq b_n\}$ and $\{n \in \mathbb{N} : m_n \geq b_n\}$ are both finite or both infinite.*