

## EXAMINATION PAPER

Examination Session: May/June

2024

Year:

Exam Code:

MATH4281-WE01

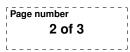
### Title:

# Topics in Combinatorics IV

Time:	3 hours	
Additional Material provided:		
Matariala Darmittadu		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators
		is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.	

Revision:

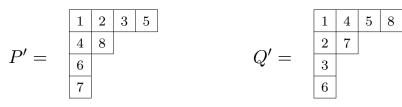


### SECTION A

- **Q1 1.1** Let *n* be a natural number, n > 3. Compute the number of Dyck paths of length 2n with the rightmost peak of height n 3.
  - **1.2** Denote by  $p_k(n)$  the number of Young diagrams  $\lambda \vdash n$  with k rows. Show that

$$p_{k-1}(n-1) + p_k(n-k) = p_k(n)$$

- **Q2** 2.1 A poset *P* is a *meet-semilattice* if every two elements have a meet. Show that a finite meet-semilattice with a unique maximal element  $\hat{1}$  is a lattice.
  - **2.2** Let  $\Delta$  be a root system of type  $C_3$ . Let  $\Delta_s$  be the set of short roots of  $\Delta$ . Show that  $\Delta_s$  is a root system and find its type.
- Q3 (a) Let  $w = 36748215 \in S_8$ . Apply the Robinson-Shensted-Knuth (RSK) algorithm to compute the insertion and recording tableaux P and Q.
  - (b) Let (P', Q') be standard Young tableaux of shape  $\lambda = (4, 2, 1, 1) \vdash 8$ , where

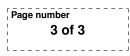


Find  $w' \in S_8$  which is taken to the pair (P', Q') by the RSK algorithm.

- **Q4** (a) Let (G, S) be a Coxeter system of type  $G_2$ . Show that two simple reflections of G are not conjugated to each other in G, i.e.  $s_1 \neq gs_2g^{-1}$  for any  $g \in G$ .
  - (b) Let (G, S) be an irreducible Coxeter system,  $G = \langle s_1, \ldots, s_n | s_i^2, (s_i s_j)^{m_{ij}} \rangle$ . Suppose that for any  $s_i, s_j \in S$  either  $m_{ij} = 2$  or  $m_{ij} = 3$ . Show that any two reflections in (G, S) are conjugated to each other.

### SECTION B

- **Q5** Given a permutation  $w = w_1 \dots w_n \in S_n$ , recall that  $i \in [n]$  is an excedance of w if  $i < w_i, i \in [n]$  is a weak excedance of w if  $i \le w_i$ , and  $i \in [n-1]$  is a descent of w if  $w_i > w_{i+1}$ . Denote by exc (w) and wexc (w) the numbers of excedances and weak excedances of w respectively.
  - (a) Compute the generating function  $f_{\text{wexc}}(x) = \sum_{w \in S_n} x^{\text{wexc}(w)}$  for n = 3.
  - (b) Recall from lectures that there exists a bijection  $f : S_n \to S_n$  taking excedances to descents. Let  $w \in S_n$  have k + 1 weak excedances, and let  $v = v_1 v_2 \dots v_n = f(w^{-1})$ . Compute the number of descents of the permutation  $v' = v_n v_{n-1} \dots v_2 v_1$ .
  - (c) Show that statistics exc and wexc -1 are equidistributed (i.e., for any k the number of permutations in  $S_n$  with k excedances is equal to the number of permutations in  $S_n$  with k + 1 weak excedances).



- **Q6** Let (G, S) be a Coxeter system. Given  $s \in S$ , denote by  $P_s$  the set of  $g \in G$  such that l(sg) > l(g), where l(g) denotes the length of g.
  - (a) Show that  $\bigcap_{s \in S} P_s = \{e\}.$
  - (b) Show that for  $s \in S$  and  $g \in G$  either l(sg) > l(g) or l(sg) < l(g).
  - (c) Show that for every  $s \in S$  the sets  $P_s$  and  $sP_s$  do not intersect, and  $P_s \cup sP_s = G$ .
  - (d) Let  $s, t \in S, g \in G$ . Show that if  $g \in P_s$  and  $gt \notin P_s$ , then sg = gt.
- **Q7** Denote by  $F_k(n)$  the number of plane trees with n edges such that the root has precisely k children.
  - (a) Show that  $F_1(n) = F_2(n) = C_{n-1}$ , the (n-1)-st Catalan number (you can use all results from lectures).
  - (b) Fix  $k \ge 1$ . Compute the generating function  $F_k(x) = \sum_{n=0}^{\infty} F_k(n) x^n$  (express  $F_k(x)$  via C(x), the generating function for Catalan numbers).
  - (c) Define  $F_0(0) = 1$  and  $F_0(n) = 0$  for n > 0. Show that the generating function  $F(x, y) = \sum_{k,n=0}^{\infty} F_k(n) x^n y^k$  is equal to  $\frac{1}{1 xyC(x)}$ .
- **Q8** Let  $\Delta$  be a root system of type  $B_3$ .
  - (a) Compute the Coxeter number of  $\Delta$  and the exponents of the Weyl group of  $\Delta$ .
  - (b) Let P be the root poset of  $\Delta$ . Draw the Hasse diagram of P.
  - (c) Draw the Hasse diagram of the poset of order ideals of P. Identify join-irreducible elements.