Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4281-WE01

Title: Topics in Combinatorics IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

SECTION A

Q1 **1.1** Let n be a natural number, $n > 3$. Compute the number of Dyck paths of length $2n$ with the rightmost peak of height $n-3$.

1.2 Denote by $p_k(n)$ the number of Young diagrams $\lambda \vdash n$ with k rows. Show that

$$p_{k-1}(n-1) + p_k(n-k) = p_k(n)$$

Q2 **2.1** A poset P is a *meet-semilattice* if every two elements have a meet. Show that a finite meet-semilattice with a unique maximal element $\hat{1}$ is a lattice.

2.2 Let Δ be a root system of type C_3. Let Δ_s be the set of short roots of Δ. Show that Δ_s is a root system and find its type.

Q3 (a) Let $w = 3\,6\,7\,4\,8\,2\,1\,5 \in S_8$. Apply the Robinson-Shensted-Knuth (RSK) algorithm to compute the insertion and recording tableaux P and Q.

(b) Let (P', Q') be standard Young tableaux of shape $\lambda = (4,2,1,1) \vdash 8$, where

$$P' = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 8 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \qquad Q' = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 8 \\ \hline 2 & 7 \\ \cline{1-2} 3 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}$$

Find $w' \in S_8$ which is taken to the pair (P', Q') by the RSK algorithm.

Q4 (a) Let (G, S) be a Coxeter system of type G_2. Show that two simple reflections of G are not conjugated to each other in G, i.e. $s_1 \neq g s_2 g^{-1}$ for any $g \in G$.

(b) Let (G, S) be an irreducible Coxeter system, $G = \langle s_1, \ldots, s_n \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$. Suppose that for any $s_i, s_j \in S$ either $m_{ij} = 2$ or $m_{ij} = 3$. Show that any two reflections in (G, S) are conjugated to each other.

SECTION B

Q5 Given a permutation $w = w_1 \ldots w_n \in S_n$, recall that $i \in [n]$ is an *excedance* of w if $i < w_i$, $i \in [n]$ is a *weak excedance* of w if $i \leq w_i$, and $i \in [n-1]$ is a *descent* of w if $w_i > w_{i+1}$. Denote by $\mathrm{exc}\,(w)$ and $\mathrm{wexc}\,(w)$ the numbers of excedances and weak excedances of w respectively.

(a) Compute the generating function $f_{\mathrm{wexc}}(x) = \sum\limits_{w \in S_n} x^{\mathrm{wexc}\,(w)}$ for $n = 3$.

(b) Recall from lectures that there exists a bijection $f : S_n \to S_n$ taking excedances to descents. Let $w \in S_n$ have $k+1$ weak excedances, and let $v = v_1 v_2 \ldots v_n = f(w^{-1})$. Compute the number of descents of the permutation $v' = v_n v_{n-1} \ldots v_2 v_1$.

(c) Show that statistics exc and wexc $-\,1$ are equidistributed (i.e., for any k the number of permutations in S_n with k excedances is equal to the number of permutations in S_n with $k+1$ weak excedances).

Q6 Let (G, S) be a Coxeter system. Given $s \in S$, denote by P_s the set of $g \in G$ such that $l(sg) > l(g)$, where $l(g)$ denotes the length of g.

(a) Show that $\bigcap_{s \in S} P_s = \{e\}$.

(b) Show that for $s \in S$ and $g \in G$ either $l(sg) > l(g)$ or $l(sg) < l(g)$.

(c) Show that for every $s \in S$ the sets P_s and sP_s do not intersect, and $P_s \cup sP_s = G$.

(d) Let $s, t \in S$, $g \in G$. Show that if $g \in P_s$ and $gt \notin P_s$, then $sg = gt$.

Q7 Denote by $F_k(n)$ the number of plane trees with n edges such that the root has precisely k children.

(a) Show that $F_1(n) = F_2(n) = C_{n-1}$, the $(n-1)$-st Catalan number (you can use all results from lectures).

(b) Fix $k \geq 1$. Compute the generating function $F_k(x) = \sum_{n=0}^{\infty} F_k(n) x^n$ (express $F_k(x)$ via $C(x)$, the generating function for Catalan numbers).

(c) Define $F_0(0) = 1$ and $F_0(n) = 0$ for $n > 0$. Show that the generating function $F(x, y) = \sum_{k,n=0}^{\infty} F_k(n) x^n y^k$ is equal to $\dfrac{1}{1 - xyC(x)}$.

Q8 Let Δ be a root system of type B_3.

(a) Compute the Coxeter number of Δ and the exponents of the Weyl group of Δ.

(b) Let P be the root poset of Δ. Draw the Hasse diagram of P.

(c) Draw the Hasse diagram of the poset of order ideals of P. Identify join-irreducible elements.