Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH42920-WE01

Title:

Functional Analysis and Applications V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2024

SECTION A

Q1 Consider the set

$$C^1[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuously differentiable on } [0,1]\}$$

together with the pointwise addition and scalar multiplication. You may use without proof that $C^1[0,1]$ is a vector space under these operations. Define

$$\|\cdot\| : C^1[0,1] \to \mathbb{R}$$

by

$$\|f\| = \|f\|_\infty + 5\|f'\|_\infty$$

where

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|.$$

1.1 Show that $\|\cdot\|$ is a norm on $C^1[0,1]$.

1.2 Show that $(C^1[0,1], \|\cdot\|)$ is a Banach space.

Q2 Let $\mathcal{X}$ and $\mathcal{Y}$ be two normed spaces and let $T \in B(\mathcal{X}, \mathcal{Y})$. Assume that for any $x, y \in \mathcal{X}$

$$\|Tx - Ty\|_{\mathcal{Y}} = \|x - y\|_{\mathcal{X}}.$$

2.1 Show that if $\mathcal{X}$ is a Banach space then so is $\mathcal{R}(T)$.

2.2 Show that if $\mathcal{R}(T)$ is separable then so is $\mathcal{X}$.

Q3 In $\mathcal{X} = C(\mathbb{R})$ equipped with $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$, define $T : \mathcal{D}(T) \subset \mathcal{X} \to \mathcal{X}$ by

$$(Tf)(x) = f'(0), \quad x \in \mathbb{R}, \quad \text{for} \quad f \in \mathcal{D}(T) = C^1(\mathbb{R}).$$

3.1 Show that the operator T is not bounded.

3.2 Show that T is not closed.
Hint: Find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ such that $\lim_{n\to\infty} f_n = 0$, $\lim_{n\to\infty} Tf_n = g$ for some $0 \neq g \in \mathcal{X}$.

Q4 In $\mathcal{H} = L^2([0,1])$ define $T : \mathcal{H} \to \mathcal{H}$ by

$$(Tf)(x) = \mathrm{i}\int_0^1 (x-y)f(y)\,\mathrm{d}y, \quad x \in [0,1], \quad \text{for} \quad f \in \mathcal{H}.$$

4.1 Show that the operator T is bounded.

4.2 Show that T is symmetric.

4.3 Show that T is selfadjoint.

SECTION B

Q5 Let $\mathcal{X}$ be a Banach space and let $T \in B(\mathcal{X}, \mathcal{X})$. Assume that there exists a non-negative sequence $\{c_n\}_{n\in\mathbb{N}}$ such that

$$\sum_{n\in\mathbb{N}} c_n < \infty$$

and for any $x, y \in \mathcal{X}$

$$\|T^n x - T^n y\| \leq c_n \|x - y\|.$$

5.1 Show that $\|T^n\| \leq c_n$.

5.2 Show that $S_N = \sum_{n=0}^{N} T^n$ converges with respect to the operator norm in $B(\mathcal{X}, \mathcal{X})$ to some bounded linear operator S.

5.3 Show that $TS = ST$.
You may use without proof the fact that if $\{A_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X}, \mathcal{X})$ converges with respect to the operator norm to $A \in B(\mathcal{X}, \mathcal{X})$ then for any $B \in B(\mathcal{X}, \mathcal{X})$ we have that

$$\lim_{n\to\infty} A_n B = AB, \qquad \text{and} \qquad \lim_{n\to\infty} BA_n = BA.$$

5.4 Show that $STx + x = Sx$ for any $x \in \mathcal{X}$. Conclude that S is invertible and that its inverse is a bounded linear operator.

Q6 Consider the operator $T : \ell_\infty(\mathbb{N}) \to \ell_\infty(\mathbb{N})$ defined by

$$T(a_1, a_2, \dots, a_n, \dots) = \left(a_1, \frac{a_1 + a_2}{2}, \dots, \frac{\sum_{i=1}^{n} a_i}{n}, \dots\right).$$

6.1 Show that T is well defined by showing that for any $\boldsymbol{a} \in \ell_\infty(\mathbb{N})$

$$\sup_{k\in\mathbb{N}} |(T\boldsymbol{a})_k| \leq \|\boldsymbol{a}\|_\infty.$$

6.2 Show that T is injective.

6.3 Explicitly find $\boldsymbol{a}_n \in \ell_\infty(\mathbb{N})$ such that $T\boldsymbol{a}_n = \boldsymbol{e}_n$ where

$$(\boldsymbol{e}_n)_k = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

Use this to show that $T^{-1} : \mathcal{R}(T) \to \ell_\infty(\mathbb{N})$ is not a bounded operator.

Q7 In $\mathcal{H} = \ell_2(\mathbb{N})$ let $T : \mathcal{H} \to \mathcal{H}$ be the linear operator defined by

$$T(x_1, x_2, x_3, \dots) = (0, 0, x_1, x_2, x_3, \dots).$$

7.1 Show that T is bounded and find $\|T\|$.

7.2 Find the adjoint operator T^* and its operator domain.

7.3 Find $\sigma_p(T)$, $\sigma_p(T^*)$ and $\sigma(T)$.

Q8 Let $\mathcal{H} = \{\boldsymbol{x} \in \ell_2(\mathbb{N}) \mid \sum_{n=1}^{\infty} |nx_n|^2 < \infty\}$. You may use without proof that $\mathcal{H}$ is a Hilbert space with scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{n=1}^{\infty} n^2 x_n \overline{y_n}$. Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be defined by

$$B(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} (\mathrm{i}^n + 2n^2) x_n \overline{y_n}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}.$$

8.1 Show that B is a sesquilinear form.

8.2 Show that the sesquilinear form B is bounded and coercive.

8.3 Let $f : \mathcal{H} \to \mathbb{C}$ be defined by

$$f(\boldsymbol{x}) = \sum_{n=1}^{\infty} x_n, \quad \boldsymbol{x} \in \mathcal{H}.$$

Show that $f \in \mathcal{H}^*$ and conclude that there exists $\boldsymbol{y} \in \mathcal{H}$ such that $B(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{H}$.