

EXAMINATION PAPER

Examination Session: May/June

2024

Year:

Exam Code:

MATH43320-WE01

Title:

Ergodic Theory V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.		

Revision:



SECTION A

- **Q1** Let (X, T) be a topological dynamical system and suppose that X does not have isolated points—in other words, for all $\varepsilon > 0$ and each $x \in X$, the ε -ball $B_{\varepsilon}(x)$ around x contains infinitely many points.
 - (a) Define what it means for (X, T) to be forward transitive.
 - (b) Show that if (X, T) is forward transitive, then for each pair of open sets $U, V \subseteq X$ there is $n \in \mathbb{N}$ with $T^n U \cap V \neq \emptyset$.
- **Q2** In the following, we consider the system (\mathbb{T}^1, E_2) , where E_2 denotes the doubling map $E_2(x) = 2x \mod 1$ ($x \in \mathbb{T}^1$).
 - (a) Show that Lebesgue measure λ is invariant for (\mathbb{T}^1, E_2) .
 - (b) Show that Lebesgue measure λ is ergodic for (\mathbb{T}^1, E_2) . *Hint:* You may use without proof that if $f \in L_2(\mathbb{T}^1, \mathcal{B}, \lambda)$ with $f(x) = \sum_{n \in \mathbb{Z}} a_n \exp(2\pi i n x)$, then $a_n \to 0$ as $|n| \to \infty$.
- **Q3** Let A be the following matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

and let $\sigma: \Sigma_A^+ \to \Sigma_A^+$ be the shift map.

- (a) Show that A is primitive.
- (b) State the Perron-Frobenius theorem for non-negative primitive matrices.
- (c) Use the fact that $\begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$ is an eigenvector of A to show that $h_{\text{top}}(\sigma) = 1$.
- **Q4** Let $T: X \to X$ be a continuous map and X a compact metric space with metric $d(\cdot, \cdot)$.
 - (a) Show that for each $n \in \mathbb{N}$,

$$d_n(x,y) := \max_{0 \le k \le n-1} d(T^k x, T^k y)$$

(for all $x, y \in X$) is a metric on X.

- (b) State the definition of the topological entropy of T in terms of the covering number of X with respect to d_n .
- (c) Show that if T is an isometry of X with respect to the metric $d(\cdot, \cdot)$, then $h_{\text{top}}(T) = 0$.

SECTION B

Q5 Let $\mathcal{A} = \{a_1, \ldots, a_m\}$ be a finite alphabet and consider $\mathcal{A}^{\mathbb{N}}$ equipped with the metric

$$d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) := 2^{-\min\{n \in \mathbb{N} \colon x_n \neq y_n\}}$$

In the following, (X, σ) denotes a subshift of $(\mathcal{A}^{\mathbb{N}}, \sigma)$ which admits only one invariant measure which we denote by μ .

Recall that (X, σ) is a subshift of $(\mathcal{A}^{\mathbb{N}}, \sigma)$ if $X \subseteq \mathcal{A}^{\mathbb{N}}$ is closed and σ -invariant, where σ denotes the left-shift on $\mathcal{A}^{\mathbb{N}}$.

- (a) Show that μ is ergodic.
- (b) Show that for each $m \in \mathbb{N}$ and each cylinder set $[u_1, \ldots, u_m]$, there is $c_{[u_1, \ldots, u_m]} \ge 0$ such that for all $x \in X$

$$\lim_{n \to \infty} \frac{1}{n} \cdot \#\{\ell = 0, \dots, n-1 \colon \sigma^{\ell} x \in [u_1, \dots, u_m]\} = c_{[u_1, \dots, u_m]}.$$

(c) With the notation of the previous item and with $\mathcal{A} = \{0, 1\}$, suppose $c_{[0]} = 1/3$ and $c_{[1]} = 1/7$. Consider the function f on X with

$$f(x) := \begin{cases} 1 & \text{if } x_1 = 0; \\ 2 & \text{if } x_1 = 1, \end{cases}$$

where $x = (x_1, x_2, \ldots)$. Compute the integral $\int_X f d\mu$.

- **Q6** In all of the following, (X, f) and (Y, g) are topological dynamical systems. We call (X, f) **positively expansive** if there is $\delta > 0$ such that for each pair $x \neq y \in X$ there is $n \in \mathbb{N} \cup \{0\}$ with $d(f^n x, f^n y) > \delta$.
 - (a) Show that the doubling map is positively expansive.
 - (b) Define what it means for (X, f) and (Y, g) to be topologically conjugate.
 - (c) Suppose (X, f) is positively expansive and (Y, g) is topologically conjugate to (X, f). Show that (Y, g) is also positively expansive.
 - (d) Now, suppose that (Y,g) is positively expansive and (X, f) is an extension of (Y,g). Is (X, f) necessarily positively expansive? Justify you answer. Hint: Think of products.
- **Q7** Let (X, \mathcal{B}, μ) be a Borel probability space and $\alpha = \{A_1, A_2, \ldots, A_n\}$ a finite measurable partition of X.
 - (a) Give the formula for the measure-theoretic entropy $H_{\mu}(\alpha)$ of the partition α .
 - (b) Use the fact that $\varphi(x) = -x \log_2 x$ is concave to show that

$$H_{\mu}(\alpha) \le \log_2 n.$$

(c) Let $\beta = \{B_1, B_2, \dots, B_m\}$ be a finite measurable partition of X that is independent of α . Show that

$$H_{\mu}(\alpha \lor \beta) = H_{\mu}(\alpha) + H_{\mu}(\beta).$$



- **Q8** Let $M \in \pm \mathrm{SL}(d, \mathbb{Z})$ be hyperbolic and let $\widetilde{M} : \mathbb{T}^d \to \mathbb{T}^d$ be the toral automorphism associated to M.
 - (a) Show that for any $n \in \mathbb{Z}_{>0}$, $M^n I$ is invertible, where I is the $d \times d$ identity matrix. *Hint:* $x^n 1 = \prod_{j=0}^{n-1} (x \omega^j)$, where $\omega = e^{2\pi i/n}$.
 - (b) Let $\mathbf{x} \in \mathbb{R}^d$ be such that $M^n \mathbf{x} = \mathbf{x} + \mathbf{j}$ for some $\mathbf{j} \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_{>0}$. Show that $\mathbf{x} \in \mathbb{Q}^d$.
 - (c) Let $\mathbf{v} \in \mathbb{Q}^d$. Writing $\mathbf{v} = \frac{1}{q}\mathbf{j}$ for some $\mathbf{j} \in \mathbb{Z}^d$ and $q \in \mathbb{Z}_{>0}$, use the fact that there are only finitely many points in $[0,1)^d \cap \frac{1}{q}\mathbb{Z}^d$ to show that $[\mathbf{v}]$ is \widetilde{M} -periodic.