Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4361-WE01

Title:
Ergodic Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Let (X, T) be a topological dynamical system and suppose that X does not have isolated points—in other words, for all $\varepsilon > 0$ and each $x \in X$, the ε-ball $B_\varepsilon(x)$ around x contains infinitely many points.

(a) Define what it means for (X, T) to be forward transitive.

(b) Show that if (X, T) is forward transitive, then for each pair of open sets $U, V \subseteq X$ there is $n \in \mathbb{N}$ with $T^n U \cap V \neq \emptyset$.

Q2 In the following, we consider the system $(\mathbb{T}^1, E_2)$, where E_2 denotes the doubling map $E_2(x) = 2x \bmod 1$ $(x \in \mathbb{T}^1)$.

(a) Show that Lebesgue measure λ is invariant for $(\mathbb{T}^1, E_2)$.

(b) Show that Lebesgue measure λ is ergodic for $(\mathbb{T}^1, E_2)$.
Hint: You may use without proof that if $f \in L_2(\mathbb{T}^1, \mathcal{B}, \lambda)$ with $f(x) = \sum_{n\in\mathbb{Z}} a_n \exp(2\pi i n x)$, then $a_n \to 0$ as $|n| \to \infty$.

Q3 Let A be the following matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

and let $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be the shift map.

(a) Show that A is primitive.

(b) State the Perron-Frobenius theorem for non-negative primitive matrices.

(c) Use the fact that $\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ is an eigenvector of A to show that $h_{\text{top}}(\sigma) = 1$.

Q4 Let $T : X \to X$ be a continuous map and X a compact metric space with metric $d(\cdot, \cdot)$.

(a) Show that for each $n \in \mathbb{N}$,

$$d_n(x, y) := \max_{0 \le k \le n-1} d(T^k x, T^k y)$$

(for all $x, y \in X$) is a metric on X.

(b) State the definition of the topological entropy of T in terms of the covering number of X with respect to d_n.

(c) Show that if T is an isometry of X with respect to the metric $d(\cdot, \cdot)$, then $h_{\text{top}}(T) = 0$.

SECTION B

Q5 Let $\mathcal{A} = \{a_1, \ldots, a_m\}$ be a finite alphabet and consider $\mathcal{A}^{\mathbb{N}}$ equipped with the metric

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) := 2^{-\min\{n\in\mathbb{N}:\, x_n \neq y_n\}}$$

In the following, (X, σ) denotes a subshift of $(\mathcal{A}^{\mathbb{N}}, \sigma)$ which admits only one invariant measure which we denote by μ.

Recall that (X, σ) is a subshift of $(\mathcal{A}^{\mathbb{N}}, \sigma)$ if $X \subseteq \mathcal{A}^{\mathbb{N}}$ is closed and σ-invariant, where σ denotes the left-shift on $\mathcal{A}^{\mathbb{N}}$.

(a) Show that μ is ergodic.

(b) Show that for each $m \in \mathbb{N}$ and each cylinder set $[u_1, \ldots, u_m]$, there is $c_{[u_1,\ldots,u_m]} \geq 0$ such that for all $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} \cdot \#\{\ell = 0, \ldots, n-1 \colon \sigma^\ell x \in [u_1, \ldots, u_m]\} = c_{[u_1,\ldots,u_m]}.$$

(c) With the notation of the previous item and with $\mathcal{A} = \{0, 1\}$, suppose $c_{[0]} = 1/3$ and $c_{[1]} = 1/7$. Consider the function f on X with

$$f(x) := \begin{cases} 1 & \text{if } x_1 = 0; \\ 2 & \text{if } x_1 = 1, \end{cases}$$

where $x = (x_1, x_2, \ldots)$. Compute the integral $\int_X f\, d\mu$.

Q6 In all of the following, (X, f) and (Y, g) are topological dynamical systems. We call (X, f) **positively expansive** if there is $\delta > 0$ such that for each pair $x \neq y \in X$ there is $n \in \mathbb{N} \cup \{0\}$ with $d(f^n x, f^n y) > \delta$.

(a) Show that the doubling map is positively expansive.

(b) Define what it means for (X, f) and (Y, g) to be topologically conjugate.

(c) Suppose (X, f) is positively expansive and (Y, g) is topologically conjugate to (X, f). Show that (Y, g) is also positively expansive.

(d) Now, suppose that (Y, g) is positively expansive and (X, f) is an extension of (Y, g). Is (X, f) necessarily positively expansive? Justify you answer.
Hint: Think of products.

Q7 Let $(X, \mathcal{B}, \mu)$ be a Borel probability space and $\alpha = \{A_1, A_2, \ldots, A_n\}$ a finite measurable partition of X.

(a) Give the formula for the measure-theoretic entropy $H_\mu(\alpha)$ of the partition α.

(b) Use the fact that $\varphi(x) = -x \log_2 x$ is concave to show that

$$H_\mu(\alpha) \leq \log_2 n.$$

(c) Let $\beta = \{B_1, B_2, \ldots, B_m\}$ be a finite measurable partition of X that is independent of α. Show that

$$H_\mu(\alpha \vee \beta) = H_\mu(\alpha) + H_\mu(\beta).$$

Q8 Let $M \in \pm\mathrm{SL}(d,\mathbb{Z})$ be hyperbolic and let $\widetilde{M} : \mathbb{T}^d \to \mathbb{T}^d$ be the toral automorphism associated to M.

(a) Show that for any $n \in \mathbb{Z}_{>0}$, $M^n - I$ is invertible, where I is the $d \times d$ identity matrix. *Hint:* $x^n - 1 = \prod_{j=0}^{n-1}(x - \omega^j)$, *where* $\omega = e^{2\pi i/n}$.

(b) Let $\mathbf{x} \in \mathbb{R}^d$ be such that $M^n\mathbf{x} = \mathbf{x} + \mathbf{j}$ for some $\mathbf{j} \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_{>0}$. Show that $\mathbf{x} \in \mathbb{Q}^d$.

(c) Let $\mathbf{v} \in \mathbb{Q}^d$. Writing $\mathbf{v} = \frac{1}{q}\mathbf{j}$ for some $\mathbf{j} \in \mathbb{Z}^d$ and $q \in \mathbb{Z}_{>0}$, use the fact that there are only finitely many points in $[0,1)^d \cap \frac{1}{q}\mathbb{Z}^d$ to show that $[\mathbf{v}]$ is $\widetilde{M}$-periodic.

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