

# **EXAMINATION PAPER**

Examination Session: May/June

2024

Year:

Exam Code:

MATH43720-WE01

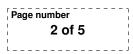
### Title:

# Stochastic Analysis V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:



#### SECTION A

**Q1** Fix real constants  $\mu < 0$  and  $\sigma^2 > 0$ . Let  $\zeta_1, \zeta_2, \ldots$  be independent, identically distributed  $\mathcal{N}(\mu, \sigma^2)$  random variables.

Let  $S_0 := 0$  and  $S_n := \sum_{i=1}^n \zeta_i$  for  $n \in \mathbb{N}$ . Also consider

$$S^\star := \sup_{n \in \mathbb{Z}_+} S_n.$$

- (a) Show that there is a unique  $\lambda > 0$  (which you should identify as a function of  $\mu$  and  $\sigma^2$ ) such that  $M_n := e^{\lambda S_n}$  defines a discrete-time martingale with respect to the natural filtration associated with  $S_n$ .
- (b) For  $y \in \mathbb{Z}_+$ , consider the stopping time  $\tau_y := \inf\{n \in \mathbb{Z}_+ : S_n > y\}$ . Using the optional stopping theorem and the martingale identified in part (a), prove that

$$1 \ge e^{\lambda y} \mathbb{P}(\tau_y \le n), \text{ for all } n \in \mathbb{Z}_+,$$

where  $\lambda$  is the constant from part (a).

(c) Deduce that

$$\mathbb{P}(S^* > y) \leq e^{-\lambda y}$$
, for all  $y \in \mathbb{Z}_+$ .

**Q2** Let  $\nu, \mu$ , and  $\rho$  be finite measures on a measurable space  $(\Omega, \mathcal{F})$ .

- (a) Explain what it means to say that  $\nu$  is absolutely continuous with respect to  $\mu$ , written  $\nu \ll \mu$ .
- (b) Show that if  $\nu \ll \mu$  and  $\mu \ll \rho$ , then  $\nu \ll \rho$ .

One version of the Radon–Nikodym theorem states that if  $\nu \ll \mu$ , then there exists a (Radon–Nikodym) derivative  $d\nu/d\mu : \Omega \to \mathbb{R}_+$  (measurable) such that

$$\int f \,\mathrm{d}\nu = \int f \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\mu \tag{1}$$

for all measurable functions  $f: \Omega \to \mathbb{R}_+$ .

- (c) Prove that the Radon–Nikodym derivative appearing in (1) is essentially unique, in a sense that you should state precisely.
- (d) Deduce that if  $\nu \ll \mu$  and  $\mu \ll \rho$ , then the corresponding derivatives satisfy

$$\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}\rho},$$

up to sets of measure zero, in a sense that you should state precisely.

- Exam code MATH43720-WE01
- **Q3** Let  $W_t, t \in \mathbb{R}_+$ , be a Brownian motion on  $\mathbb{R}$  and define the martingale  $X_t, t \in \mathbb{R}_+$ , by

$$X_t = \int_0^t \mathrm{e}^{-s} \,\mathrm{d}W_s.$$

- (a) State the definition of uniform integrability and prove that X given above is a uniformly integrable martingale by verifying the definition.
- (b) Use the down-crossing inequality to prove that  $\lim_{t\to+\infty} X_t = X_{\infty}$  exists almost surely.
- (c) State a theorem that guarantees the convergence in Part (b) also holds in  $L^1$  and use the  $L^1$  convergence to prove that

$$X_t = \mathbb{E}[X_{\infty}|\mathcal{F}_t], \text{ a.s.}$$

where  $\mathcal{F}_t = \sigma(W_s, 0 \le s \le t)$ .

Q4 (a) Using Itô's formula, prove that if  $M_t, t \in \mathbb{R}_+$ , is a continuous local martingale, and  $\lambda \in \mathbb{R}$  is a constant, then the process

$$\mathcal{E}^{\lambda}(M)_t = \exp\left(\lambda M_t - \frac{\lambda^2}{2} \langle M, M \rangle_t\right), \ t \in \mathbb{R}_+,$$

is a local martingale.

(b) If  $M_0 = 0$ , prove that  $\mathcal{E}^{\lambda}(M)$  is a martingale if and only if  $\mathbb{E}[\mathcal{E}^{\lambda}(M)_t] = 1$  for every  $t \in \mathbb{R}_+$ .

#### **SECTION B**

**Q5** Let  $W = (W_t)_{t \in \mathbb{R}_+}$  be a Brownian motion. For  $t \in \mathbb{R}_+$ , define

$$U_t := 2W_{1+(t/2)} - W_1 - W_{1+t}.$$

In your answers to the following parts, you may use, without proof, results from lectures on multivariate normal distributions and/or Brownian motion, but you should state them clearly.

- (a) Find the distribution of  $U_t$  for  $t \in \mathbb{R}_+$ .
- (b) For  $s \in \mathbb{R}_+$  and  $t \in \mathbb{R}_+$ , show that  $(W_{1+s}, U_t)$  has a bivariate normal distribution, and compute  $\mathbb{E}(W_{1+s}U_t)$ .
- (c) Deduce that, for every  $t \in \mathbb{R}_+$ ,  $U_t$  is independent of  $W_1$ , and that, for every  $t \in \mathbb{R}_+$ ,  $U_t$  is independent of  $W_{1+t}$ .
- (d) Compute  $\mathbb{E}(U_s U_t)$  for  $s, t \in \mathbb{R}_+$ .
- (e) Is  $U = (U_t)_{t \in \mathbb{R}_+}$  a Brownian motion? Justify your answer.
- **Q6** Let  $\xi_1, \xi_2, \ldots$  be a sequence of independent, identically distributed random variables with  $\mathbb{P}(\xi_i = +1) = \mathbb{P}(\xi_i = -1) = 1/2$ .

Define  $S_0 := 0$  and  $S_n := \sum_{i=1}^n \xi_i$  for  $n \in \mathbb{N}$ .

For a positive integer a, define  $\tau_a := \inf\{n \in \mathbb{Z}_+ : S_n = a\}$ .

Fix  $\theta \in \mathbb{R}$  and define

$$X_n := \frac{\mathrm{e}^{\theta S_n}}{(\cosh \theta)^n}, \text{ for all } n \in \mathbb{Z}_+.$$

- (a) Show that, for any  $\theta \in \mathbb{R}$ ,  $X = (X_n)_{n \in \mathbb{Z}_+}$  is a martingale with respect to the natural filtration associated with  $S_n$ .
- (b) For  $\theta > 0$ , prove that, almost surely,

$$\lim_{n \to \infty} X_{n \wedge \tau_a} = \frac{\mathrm{e}^{\theta a}}{(\cosh \theta)^{\tau_a}} \mathbb{1}\{\tau_a < \infty\}.$$

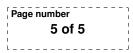
(c) Give a justification of the fact that

$$\lim_{\theta \to 0} \left( \frac{\mathrm{e}^{\theta a}}{(\cosh \theta)^{\tau_a}} \right) \mathbb{1}\{\tau_a < \infty\} = \mathbb{1}\{\tau_a < \infty\}.$$

(d) Prove that  $\mathbb{P}(\tau_a < \infty) = 1$  and, for every  $\theta \ge 0$ ,

$$\mathbb{E}\left[(\operatorname{sech}\theta)^{\tau_a}\right] = \mathrm{e}^{-\theta a},$$

where sech  $y := 1/\cosh y$ .



Q7 In your answers to the following parts, you may use, without proof, the result that any bounded and continuous martingale M is of finite quadratic variation and  $\langle M, M \rangle$  is the unique continuous, adapted and increasing process such that  $M^2 - \langle M, M \rangle$  is a martingale.

Now let  $M_t, t \in \mathbb{R}_+$ , and  $N_t, t \in \mathbb{R}_+$ , be two continuous real-valued local martingales.

Exam code

MATH43720-WE01

- (a) Prove that there is a unique continuous increasing adapted process  $\langle M, M \rangle$ , vanishing at zero, such that  $M^2 \langle M, M \rangle$  is a continuous local martingale.
- (b) Let  $\{\Delta_n\}$  be a sequence of subdivisions of  $\mathbb{R}_+$ , each with only a finite number of points in [0, t] for each  $t \ge 0$ . Prove that as  $|\Delta_n| \to 0$ , for each  $t \ge 0$

$$\sup_{s \le t} |T_s^{\Delta_n}(M) - \langle M, M \rangle_s| \to 0,$$

in probability. Here

$$T_t^{\Delta_n}(M) = \sum_{i=0}^{k-1} (M_{t_{i+1}} - M_{t_i})^2 + (M_t - M_{t_k})^2,$$

where  $\Delta_n = \{t_0 = 0 < t_1 < t_2 < \cdots\}$  and k is the unique integer such that  $t_k \leq t < t_{k+1}$ .

- (c) Prove that there is a unique continuous process  $\langle M, N \rangle$  of finite variation, vanishing at zero, such that  $MN \langle M, N \rangle$  is a local martingale.
- **Q8** Let  $W_t, t \in \mathbb{R}_+$ , be a Brownian motion on  $\mathbb{R}$  and define for any bounded measurable function  $f : \mathbb{R} \to \mathbb{R}$ ,

$$P_t f(x) = \mathbb{E}[f(x+W_t)], \quad t \in \mathbb{R}_+, x \in \mathbb{R}.$$

(a) Prove, using the Markov property, that for any  $t, s \ge 0$ , and f being bounded and measurable,

$$P_t \circ P_s f = P_{t+s} f.$$

- (b) Let A be the infinitesimal generator of  $X_t^x = x + W_t$ ,  $t \in \mathbb{R}_+$ . Prove that  $Af = \frac{1}{2} \frac{d^2}{dx^2} f$  when  $f \in C^2$  is bounded and has bounded continuous derivatives up to second order.
- (c) Prove that for any t > 0, for f being bounded and measurable,  $P_t f$  is in the domain of A and

$$\frac{\mathrm{d}}{\mathrm{d}t}P_tf = AP_tf.$$