

## **EXAMINATION PAPER**

Examination Session: May/June

2024

Year:

Exam Code:

MATH43820-WE01

Title:

Superstrings V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

**Revision:** 

## SECTION A

Q1 Consider the following classical trajectory of an open string:

$$X^{0} = \omega \tau, \ X^{1} = \cos(\omega \tau) \cos(\omega \sigma), \ X^{2} = \sin(a\tau) \cos(b\sigma),$$

and assume conformal gauge.

- 1.1 Determine the values of a and b so that the open string solves the equations of motion  $(\ddot{X} X'')^{\mu} = 0$  and the Virasoro constraints  $(\dot{X} \pm X')^2 = 0$ .
- **1.2** Recalling that the endpoints of the string are at  $\sigma = 0$  and  $\sigma = \pi$ , what constraint does  $\omega$  need to satisfy in order for the string to have Neuman boundary conditions?
- **Q2** For a closed bosonic string we may write  $X^{\mu}(z, \bar{z}) = X^{\mu}(\bar{z}) + X^{\mu}(z)$  where  $X^{\mu}(z)$  and  $X^{\mu}(\bar{z})$  have the following operator product expansions:

$$X^{\mu}(z)X^{\nu}(w) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-w) + \dots$$
 (1)

$$X^{\mu}(\bar{z})X^{\nu}(\bar{w}) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(\bar{z}-\bar{w}) + \dots$$
 (2)

$$X^{\mu}(z)X^{\nu}(\bar{w}) = ...,$$
 (3)

where ... denote non-singular terms in the expansion.

**2.1** Use (1)-(3) to derive the following operator product expansions:

$$T(z)X^{\mu}(w,\bar{w}) = \frac{1}{z-w}\partial X^{\mu}(w,\bar{w}) + \dots, \quad \tilde{T}(\bar{z})X^{\mu}(w,\bar{w}) = \frac{1}{\bar{z}-\bar{w}}\bar{\partial}X^{\mu}(w,\bar{w}) + \dots$$
(4)

where

$$T(z) = -\frac{1}{\alpha'} : \partial X(z) \cdot \partial X(z) :, \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X(\bar{z}) \cdot \bar{\partial} X(\bar{z}) : .$$

**2.2** Use (4) to compute the singular part of the operator product expansion of T and  $\tilde{T}$  with the following operators:

$$\partial X^{\mu}(w,\bar{w}), \ \bar{\partial} X^{\mu}(w,\bar{w}), \ \partial^2 X^{\mu}(w,\bar{w}).$$

Which of these operators are conformal primaries? What are their conformal weights?

Q3 Consider the Clifford algebra

$$\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2\,\eta^{\mu\nu}\,\mathbb{I}\,,$$

with I the identity operator and  $\eta^{\mu\nu} = \text{diag}(-1, 1, ..., 1)$  the standard *D*-dimensional Minkowski metric. Show that in any unitary representation of the Clifford algebra, i.e. any representation such that  $(\Gamma^{\mu})^{\dagger} = (\Gamma^{\mu})^{-1}$ , we must have

$$(\Gamma^0)^{\dagger} = -\Gamma^0, \qquad (\Gamma^i)^{\dagger} = \Gamma^i, \ i = 1, ..., D - 1.$$

Then deduce the identity

$$(\Gamma^{\mu})^{\dagger} = \Gamma^0 \Gamma^{\mu} \Gamma^0 \,.$$

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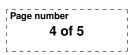
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Page number	Exam code
3 of 5	MATH43820-WE01
1	

**Q4** Consider a Grassmann operator  $\hat{Q}$  which is the generator of a fermionic symmetry. On any field operator  $\hat{\Phi}$  (bosonic or fermionic) the infinitesimal symmetry transformation generated by  $\hat{Q}$  is given by  $\delta \hat{\Phi} = i[\epsilon \hat{Q}, \hat{\Phi}]$  with  $\epsilon$  an infinitesimal Grassmann parameter.

Show that

$$\delta \hat{\Phi} = \begin{cases} i\epsilon [\hat{Q}, \hat{\Phi}], & \text{for } \hat{\Phi} \text{ bosonic}, \\ i\epsilon \{\hat{Q}, \hat{\Phi}\}, & \text{for } \hat{\Phi} \text{ fermionic}. \end{cases}$$



## SECTION B

Q5 Recall the Nambu-Goto action

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det\left(\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}G_{\mu\nu}\right)},$$

where  $G_{\mu\nu}$  is the background metric. Consider a closed string in a background metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2.$$

**5.1** Let  $t = \tau$  and  $\theta = \sigma$ , and  $r = r(\tau)$ . Show that the action is given by

$$S = -\frac{1}{\alpha'} \int dt \, r \sqrt{1 - \dot{r}^2}.$$

**5.2** Let  $\mathcal{L} = -\frac{1}{\alpha'} r \sqrt{1 - \dot{r}^2}$ . Show that the Hamiltonian  $\mathcal{H} = \prod_r \dot{r} - \mathcal{L}$  can be written as

$$\mathcal{H} = \sqrt{\Pi_r^2 + V(r)},$$

where  $\Pi_r = \partial \mathcal{L} / \partial \dot{r}$  and you must find the potential V(r). Describe qualitatively how the string moves in this potential.

**Q6** The closed string scattering amplitude for n tachyons in bosonic string theory takes the form

$$\mathcal{A}_n \propto \int \prod_{i=1}^n d^2 z_i \prod_{j < l} |z_j - z_l|^{\alpha' p_j \cdot p_l},$$

where the external momenta satisfy  $\sum_{i=1}^{n} p_i^{\mu} = 0$  and  $p_i^2 = 4/\alpha'$ . Now consider an  $SL(2, \mathbb{C})$  transformation

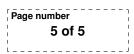
$$z_i \to \frac{az_i + b}{cz_i + d},$$

where a, b, c, d are complex numbers statisfying ad - bc = 1.

6.1 Show that under the above transformation

$$dz_i \to \frac{dz_i}{\left(cz_i + d\right)^2}$$
$$z_i - z_j \to \frac{z_i - z_j}{\left(cz_i + d\right)\left(cz_j + d\right)}$$

**6.2** Show that the *n*-point tachyon amplitude is  $SL(2, \mathbb{C})$  invariant.





Q7 Consider the quantum Hamiltonian

$$\hat{H} = \hat{a}^{\dagger}\hat{a} + \hat{f}^{\dagger}\hat{f} \,,$$

where  $\hat{a}^{\dagger}$ ,  $\hat{a}$  are canonical bosonic creation/annihilation operators and  $\hat{f}^{\dagger}$ ,  $\hat{f}$  are canonical fermionic creation/annihilation operators, i.e. they satisfy

$$\begin{split} & [\hat{a}, \hat{a}^{\dagger}] = 1, \qquad [\hat{a}, \hat{a}] = [\hat{a}^{\dagger}, \hat{a}^{\dagger}] = 0, \\ & \{\hat{f}, \hat{f}^{\dagger}\} = 1, \qquad \{\hat{f}, \hat{f}\} = \{\hat{f}^{\dagger}, \hat{f}^{\dagger}\} = 0, \end{split}$$

while all the bosonic operators commute with all the fermionic operators:

$$[\hat{a}, \hat{f}] = [\hat{a}^{\dagger}, \hat{f}] = [\hat{a}, \hat{f}^{\dagger}] = [\hat{a}^{\dagger}, \hat{f}^{\dagger}] = 0.$$

The Fock vacuum  $|0\rangle$  is the unique normalised state annihilated simultaneously by  $\hat{a}$  and  $\hat{f}$ , i.e.

$$\hat{a} \left| 0 \right\rangle = \hat{f} \left| 0 \right\rangle = 0 \,.$$

7.1 Consider the state

$$|n,0\rangle = (\hat{a}^{\dagger})^n |0\rangle ,$$

and compute  $\hat{H} | n, 0 \rangle$ .

- **7.2** Define the fermionic operator  $\hat{Q}^{\dagger} = \hat{a}\hat{f}^{\dagger}$  and compute  $\hat{Q}^{\dagger} | n, 0 \rangle$ .
- **7.3** Finally compute  $\hat{H}\hat{Q}^{\dagger}|n,0\rangle$ . From this calculation which consequence can you deduce for the spectrum of the Hamiltonian?
- Q8 The RNS gauge-fixed Polyakov action is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 \boldsymbol{\sigma} \, \eta^{ab} \left( \partial_a X^{\mu} \partial_b X^{\nu} + i \bar{\Psi}^{\mu} \rho_a \partial_b \Psi^{\nu} \right) \eta_{\mu\nu} \,,$$

where  $X^{\mu}(\boldsymbol{\sigma})$  are *D* worldsheet bosonic fields, while  $\Psi^{\mu}(\boldsymbol{\sigma})$  are *D* worldsheet Majorana fermions. The 2-dimensional Dirac matrices are

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- 8.1 Compute the equations of motions for  $X^{\mu}(\boldsymbol{\sigma})$  and  $\Psi^{\mu}(\boldsymbol{\sigma})$ . You may discard boundary terms.
- 8.2 Show that the supercurrent

$$J_a = \frac{1}{2} \rho^b \rho_a \Psi^\mu \partial_b X_\mu \,,$$

satisfies the off-shell (i.e. without using the equations of motion) relation

$$\rho^a J_a = 0$$

8.3 Show that the supercurrent  $J^a$  is conserved on-shell, i.e. show that

$$\partial^a J_a = 0$$