Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH43820-WE01

Title:
Superstrings V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Consider the following classical trajectory of an open string:

$$X^0 = \omega\tau, \quad X^1 = \cos(\omega\tau)\cos(\omega\sigma), \quad X^2 = \sin(a\tau)\cos(b\sigma),$$

and assume conformal gauge.

1.1 Determine the values of a and b so that the open string solves the equations of motion $\left(\ddot{X} - X''\right)^\mu = 0$ and the Virasoro constraints $\left(\dot{X} \pm X'\right)^2 = 0$.

1.2 Recalling that the endpoints of the string are at $\sigma = 0$ and $\sigma = \pi$, what constraint does ω need to satisfy in order for the string to have Neuman boundary conditions?

Q2 For a closed bosonic string we may write $X^\mu(z,\bar{z}) = X^\mu(\bar{z}) + X^\mu(z)$ where $X^\mu(z)$ and $X^\mu(\bar{z})$ have the following operator product expansions:

$$X^\mu(z)X^\nu(w) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-w) + ... \tag{1}$$

$$X^\mu(\bar{z})X^\nu(\bar{w}) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(\bar{z}-\bar{w}) + ... \tag{2}$$

$$X^\mu(z)X^\nu(\bar{w}) = ..., \tag{3}$$

where ... denote non-singular terms in the expansion.

2.1 Use (1)-(3) to derive the following operator product expansions:

$$T(z)X^\mu(w,\bar{w}) = \frac{1}{z-w}\partial X^\mu(w,\bar{w}) + ..., \quad \tilde{T}(\bar{z})X^\mu(w,\bar{w}) = \frac{1}{\bar{z}-\bar{w}}\bar{\partial} X^\mu(w,\bar{w}) + ... \tag{4}$$

where

$$T(z) = -\frac{1}{\alpha'} : \partial X(z)\cdot\partial X(z) :, \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X(\bar{z})\cdot\bar{\partial} X(\bar{z}) : .$$

2.2 Use (4) to compute the singular part of the operator product expansion of T and $\tilde{T}$ with the following operators:

$$\partial X^\mu(w,\bar{w}), \quad \bar{\partial} X^\mu(w,\bar{w}), \quad \partial^2 X^\mu(w,\bar{w}).$$

Which of these operators are conformal primaries? What are their conformal weights?

Q3 Consider the Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\,\eta^{\mu\nu}\,\mathbb{I},$$

with $\mathbb{I}$ the identity operator and $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$ the standard D-dimensional Minkowski metric. Show that in any unitary representation of the Clifford algebra, i.e. any representation such that $(\Gamma^\mu)^\dagger = (\Gamma^\mu)^{-1}$, we must have

$$(\Gamma^0)^\dagger = -\Gamma^0, \qquad (\Gamma^i)^\dagger = \Gamma^i, \; i = 1, ..., D-1.$$

Then deduce the identity

$$(\Gamma^\mu)^\dagger = \Gamma^0\Gamma^\mu\Gamma^0.$$

Q4 Consider a Grassmann operator $\hat{Q}$ which is the generator of a fermionic symmetry. On any field operator $\hat{\Phi}$ (bosonic or fermionic) the infinitesimal symmetry transformation generated by $\hat{Q}$ is given by $\delta\hat{\Phi} = i[\epsilon\,\hat{Q}, \hat{\Phi}]$ with ϵ an infinitesimal Grassmann parameter.

Show that

$$\delta\hat{\Phi} = \begin{cases} i\epsilon\,[\hat{Q}, \hat{\Phi}]\,, & \text{for } \hat{\Phi} \text{ bosonic}\,, \\ i\epsilon\,\{\hat{Q}, \hat{\Phi}\}\,, & \text{for } \hat{\Phi} \text{ fermionic}\,. \end{cases}$$

SECTION B

Q5 Recall the Nambu-Goto action

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}\right)},$$

where $G_{\mu\nu}$ is the background metric. Consider a closed string in a background metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2.$$

5.1 Let $t = \tau$ and $\theta = \sigma$, and $r = r(\tau)$. Show that the action is given by

$$S = -\frac{1}{\alpha'}\int dt\, r\sqrt{1-\dot{r}^2}.$$

5.2 Let $\mathcal{L} = -\frac{1}{\alpha'} r\sqrt{1-\dot{r}^2}$. Show that the Hamiltonian $\mathcal{H} = \Pi_r \dot{r} - \mathcal{L}$ can be written as

$$\mathcal{H} = \sqrt{\Pi_r^2 + V(r)},$$

where $\Pi_r = \partial\mathcal{L}/\partial\dot{r}$ and you must find the potential $V(r)$. Describe qualitatively how the string moves in this potential.

Q6 The closed string scattering amplitude for n tachyons in bosonic string theory takes the form

$$\mathcal{A}_n \propto \int \Pi_{i=1}^n d^2 z_i \Pi_{j<l} \left|z_j - z_l\right|^{\alpha' p_j \cdot p_l},$$

where the external momenta satisfy $\sum_{i=1}^n p_i^\mu = 0$ and $p_i^2 = 4/\alpha'$. Now consider an $SL(2,\mathbb{C})$ transformation

$$z_i \to \frac{az_i + b}{cz_i + d},$$

where a, b, c, d are complex numbers statisfying $ad - bc = 1$.

6.1 Show that under the above transformation

$$dz_i \to \frac{dz_i}{\left(cz_i + d\right)^2}$$

$$z_i - z_j \to \frac{z_i - z_j}{\left(cz_i + d\right)\left(cz_j + d\right)}.$$

6.2 Show that the n-point tachyon amplitude is $SL(2,\mathbb{C})$ invariant.

Q7 Consider the quantum Hamiltonian

$$\hat{H} = \hat{a}^\dagger \hat{a} + \hat{f}^\dagger \hat{f}\,,$$

where $\hat{a}^\dagger, \hat{a}$ are canonical bosonic creation/annihilation operators and $\hat{f}^\dagger, \hat{f}$ are canonical fermionic creation/annihilation operators, i.e. they satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1\,, \qquad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0\,,$$
$$\{\hat{f}, \hat{f}^\dagger\} = 1\,, \qquad \{\hat{f}, \hat{f}\} = \{\hat{f}^\dagger, \hat{f}^\dagger\} = 0\,,$$

while all the bosonic operators commute with all the fermionic operators:

$$[\hat{a}, \hat{f}] = [\hat{a}^\dagger, \hat{f}] = [\hat{a}, \hat{f}^\dagger] = [\hat{a}^\dagger, \hat{f}^\dagger] = 0\,.$$

The Fock vacuum $|0\rangle$ is the unique normalised state annihilated simultaneously by $\hat{a}$ and $\hat{f}$, i.e.

$$\hat{a}\,|0\rangle = \hat{f}\,|0\rangle = 0\,.$$

7.1 Consider the state

$$|n, 0\rangle = (\hat{a}^\dagger)^n\,|0\rangle\,,$$

and compute $\hat{H}\,|n, 0\rangle$.

7.2 Define the fermionic operator $\hat{Q}^\dagger = \hat{a}\hat{f}^\dagger$ and compute $\hat{Q}^\dagger\,|n, 0\rangle$.

7.3 Finally compute $\hat{H}\hat{Q}^\dagger\,|n, 0\rangle$. From this calculation which consequence can you deduce for the spectrum of the Hamiltonian?

Q8 The RNS gauge-fixed Polyakov action is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\boldsymbol{\sigma}\, \eta^{ab} \left(\partial_a X^\mu \partial_b X^\nu + i\bar{\Psi}^\mu \rho_a \partial_b \Psi^\nu \right) \eta_{\mu\nu}\,,$$

where $X^\mu(\boldsymbol{\sigma})$ are D worldsheet bosonic fields, while $\Psi^\mu(\boldsymbol{\sigma})$ are D worldsheet Majorana fermions. The 2-dimensional Dirac matrices are

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

8.1 Compute the equations of motions for $X^\mu(\boldsymbol{\sigma})$ and $\Psi^\mu(\boldsymbol{\sigma})$. You may discard boundary terms.

8.2 Show that the supercurrent

$$J_a = \frac{1}{2}\rho^b \rho_a \Psi^\mu \partial_b X_\mu\,,$$

satisfies the off-shell (i.e. without using the equations of motion) relation

$$\rho^a J_a = 0\,.$$

8.3 Show that the supercurrent J^a is conserved on-shell, i.e. show that

$$\partial^a J_a = 0\,.$$