EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4411-WE01

Title:

Advanced Mathematical Biology IV

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Consider the process $\{X(t) : t \geq 0\}$ obeying the following linear stochastic differential equation (written in terms of its computational definition):

$$X(t + \Delta t) = X(t) + \alpha(1 - X(t))\Delta t + \beta X(t)\sqrt{\Delta t}\,\xi, \quad t > 0, \quad X(0) = 1, \qquad (1)$$

where $\xi \sim N(0, 1)$ and $\alpha, \beta \in (0, \infty)$. Let $M(t) = \mathbb{E}[X(t)]$ and $S(t) = \mathbb{E}[X(t)^2]$.

By using the computational definition (1), and then letting $\Delta t \downarrow 0$, derive the ordinary differential equations (ODEs) obeyed by $M(t)$ and $S(t)$.

Q2 Suppose that molecules of two chemical species, A and B, are reacting in a container of volume ν with their reactions given by

$$\emptyset \xrightarrow{k_1} A \xrightarrow{k_2} B \xrightarrow{k_3} \emptyset, \qquad k_2 \neq k_3,$$

with $A(0) = B(0) = 0$. Let $P(n, m, t) = \mathbb{P}[A(t) = n,\ B(t) = m]$ where $A(t)$ and $B(t)$ denote the number of A and B molecules in the system at time t respectively.

2.1 Write down the chemical master equations for the process.

2.2 Let

$$M_A(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} nP(n, m, t), \quad M_B(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} mP(n, m, t)$$

denote the mean of the number of A and B molecules respectively. Use the chemical master equations to derive and solve ODEs for $M_A(t)$ and $M_B(t)$.

Q3 Consider a symmetric stenosis, that is a narrowing of a blood vessel, as shown below. For simplicity, the blood vessel is depicted in two dimensions, however you should consider that blood is flowing inside a vessel modelled as a three-dimensional cylindrical tube of length L. Assume that the velocity profile takes the form

$$u_x(r) = u_{\max}\left(1 - \frac{r^2}{R^2}\right),$$

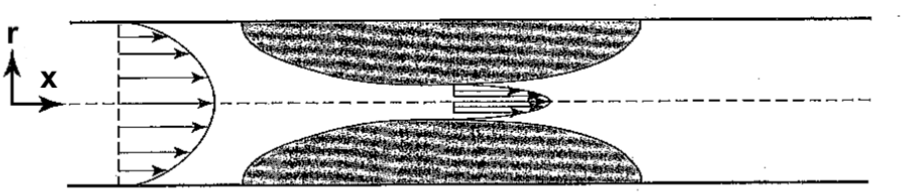
where, outside the stenosis, the radius R equals R_0 and the maximum velocity $u_{\max}$ is constant, and, within the stenosis, the radius of the vessel R equals $R_\text{i}(x)$ given by

$$R_\text{i}(x) = R_0\left\{1 - 0.5\left[1 - 4\left(\frac{x}{L}\right)^2\right]^{1/2}\right\},$$

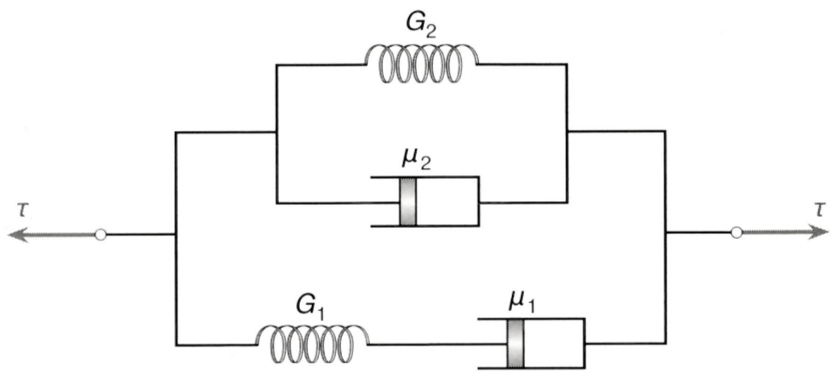
where $x \in [-L/2, L/2]$. The following holds $u_r = u_\theta = 0$.

(a) Find an expression for $u_{\max}$ in the stenosis in terms of the volume flow rate Q, the vessel radius outside the stenosis R_0, and the distance along the stenosis x/L.

(b) Compute the shear stress acting on the surface of the stenosis ($r = R_\text{i}$) at $x = 0$, assuming that the blood is a Newtonian fluid. Show that it is 8 times higher than the shear stress outside the stenosis.



Q4 Arterial walls can be modelled as viscoelastic materials whose rheological behaviour is described by the mechanical analog model shown below.



(a) State the viscoelastic elements composing the model shown above, and their connection. For each of the constituent viscoelastic elements, derive the constitutive equation.

(b) Show that the constitutive equation of the mechanical analog model takes the form

$$\mu_1\dot{\tau} + G_1\tau = \mu_1\mu_2\ddot{\gamma} + (\mu_1 G_1 + \mu_2 G_1 + \mu_1 G_2)\,\dot{\gamma} + G_1 G_2\gamma,$$

where μ_1 and μ_2 denote viscosities, G_1 and G_2 denote elastic moduli, τ and γ represent the stress and strain, respectively.

SECTION B

Q5 We are building a model for the movement of a single animal in one spatial dimension. Denote the animal's position at time $t \geq 0$ by $X(t) \in \mathbb{R}$.

5.1 Write down a stochastic differential equation (SDE) model of the form

$$X(t + \Delta t) = X(t) + f(X(t))\Delta t + g(X(t))\sqrt{\Delta t}\,\xi, \quad t \geq 0,$$

describing the animal's movement dynamics incorporating the following facts:

- the animal spends most of its time consuming food sources located at $x = 2$ and $x = 4$,
- there is an uncrossable river located at $x = 0$,
- the noise level in the model does not depend on the current position.

HINT: Specify the drift and diffusion coefficients, along with any boundary conditions. With no noise, we want stable fixed points where the animal feeds.

5.2 Write down the Fokker-Planck equation and any associated boundary conditions for your model. Hence compute the stationary distribution of the animal's position in space.

5.3 Write an expression for the average time it takes the animal to reach the food source at $x = 4$ starting from the river.

[You do not need to evaluate the resulting expression.]

5.4 (a) If we remove the food sources from the model, explain how the movement of N identical animals could be modelled using a discrete space (compartmental) approach.

(b) State one advantage and one disadvantage of this compartmental approach compared to a continuous space (SDE) model of the movement of N animals.

Q6 Suppose that two chemical species, X and Y, are reacting in a container of volume ν with their reaction dynamics given by

$$Y \xrightarrow{k_1} 2X, \quad 2X \xrightarrow{k_2} X + Y, \quad X + Y \xrightarrow{k_3} Y, \quad X \xrightarrow{k_4} \emptyset. \tag{2}$$

6.1 Write down the propensity functions for each of the reactions and the total propensity function for the process (2).

6.2 Suppose that the reaction rates of the system are given by:

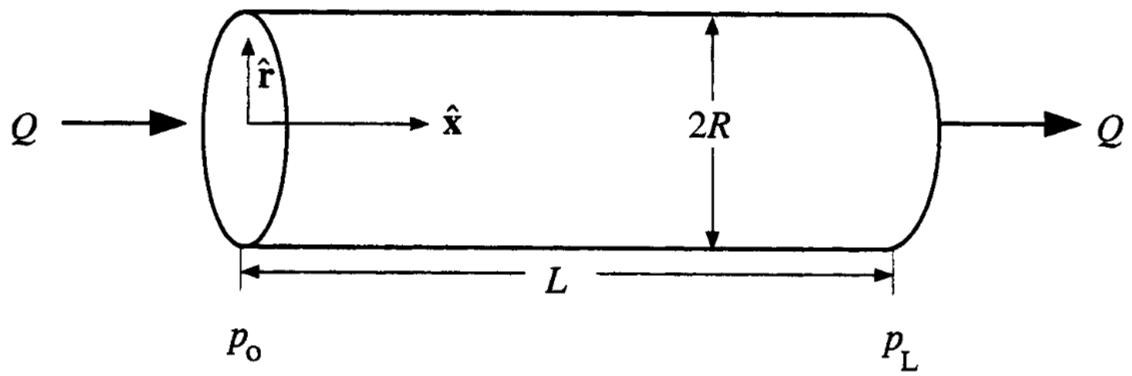
$$k_1 = 12 \text{ min}^{-1}, \quad k_2/\nu = 6 \text{ min}^{-1}, \quad k_3/\nu = 2 \text{ min}^{-1}, \quad k_4 = 8 \text{ min}^{-1}.$$

(a) Write down a deterministic model for this process based on the law of mass action and carry out a qualitative analysis of the equilibrium solutions.

(b) What would the typical dynamics look like if you simulated sample paths of the stochastic process (2) with the given parameters?

6.3 Write down a continuum approximation of the reaction process (2) in terms of the appropriate chemical Fokker-Planck equation.

Q7 Consider blood flowing through an artery. The blood is considered to be a power-law fluid, while the artery is modelled as a cylindrical tube of radius R and length L, as shown below. Assume that blood is incompressible, the blood flow is steady and laminar, and that the gravity is negligible.



(a) Show that the shear stress τ_{rx} takes the following form

$$\tau_{rx}(r) = -\left(\frac{p_0 - p_L}{2L}\right) r,$$

where p_0 and p_L are the pressures at the inlet, $x = 0$, and outlet, $x = L$, of the artery.

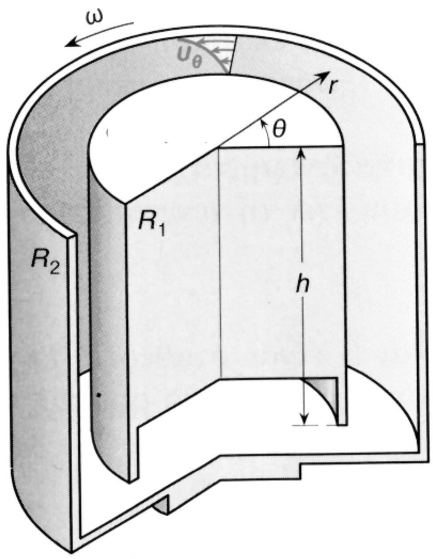
(b) Show that the velocity profile of the blood is

$$u_x(r) = \left(-\frac{p_0 - p_L}{2\kappa L}\right)^{1/n} \frac{R^{1/n+1}}{1/n+1}\left[1 - \left(\frac{r}{R}\right)^{1/n+1}\right],$$

where κ and n denote, respectively, the fluid consistency coefficient and the flow behaviour index of the power-law fluid.

(c) Given that blood exhibits shear-thinning behaviour, which of the following conditions is correct for the power-law parameter n when modelling blood: (i) $n < 1$, (ii) $n = 1$, or (iii) $n > 1$? Make a qualitative plot of the velocity profile for a power-law fluid flowing inside a cylindrical tube, and compare it to that of a Newtonian fluid.

Q8 A Couette rheometer, as shown below, is used to measure the viscosity of healthy human blood plasma at room temperature. The outer cylinder of radius $R_2 = 5.2$ mm is rotated at a constant angular velocity $\omega = 2$ rad/s. The inner cylinder, which is stationary, has a radius of $R_1 = 5.0$ mm, while the height of the rheometer is $h = 1.0$ mm. The azimuthal velocity, u_θ, of blood plasma is assumed to vary linearly with the distance r from the central axis of the rheometer.



(a) Show that the azimuthal velocity can be expressed as

$$u_\theta(r) = \frac{\omega R_2}{R_2 - R_1}(r - R_1).$$

(b) Assume that blood plasma behaves like a Newtonian fluid. Show that the force F exerted by the blood plasma on the outer cylinder is given by

$$F = \frac{2\pi\mu\omega R_2^2 h}{R_2 - R_1}.$$

(c) If the torque is measured to be $\mathcal{M} = 10.0\ \mathrm{N}\cdot\mu\mathrm{m}$, find the viscosity μ of the blood plasma.

(d) In the above analysis, we have assumed that the blood plasma under examination behaves like a Newtonian fluid. Is this assumption correct, considering the specific rheometer setup and given that blood plasma behaves like a Newtonian fluid for shear rates $< 100\ \mathrm{s}^{-1}$? Justify your answer. If this assumption is not correct, which non-Newtonian fluid models would be better to consider?