Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH4421-WE01

Title: Geophysical and Astrophysical Fluids IV

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 Consider an inertial frame defined by the orthogonal basis $\boldsymbol{e}_i = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and a rotating frame with the orthogonal basis $\hat{\boldsymbol{e}}_i = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$. The axis of rotation for the rotating frame is aligned with $\hat{\boldsymbol{e}}_2 = 1/2\boldsymbol{e}_2 + \sqrt{3}/2\boldsymbol{e}_3$ and rotates with a constant angular velocity of $\Omega = 1$. The vectors $\boldsymbol{e}_1$ and $\hat{\boldsymbol{e}}_1$ are initially aligned (only at $t = 0$). It can be shown that the unit vectors of these two frames are related to each other as below

$$\boldsymbol{e}_1 = \cos(t)\hat{\boldsymbol{e}}_1 + \sin(t)\hat{\boldsymbol{e}}_3, \tag{1a}$$

$$\boldsymbol{e}_2 = \frac{\sqrt{3}}{2}\sin(t)\hat{\boldsymbol{e}}_1 + \frac{1}{2}\hat{\boldsymbol{e}}_2 - \frac{\sqrt{3}}{2}\cos(t)\hat{\boldsymbol{e}}_3, \tag{1b}$$

$$\boldsymbol{e}_3 = \frac{1}{2}\cos(t)\hat{\boldsymbol{e}}_3 - \frac{1}{2}\sin(t)\hat{\boldsymbol{e}}_1 + \frac{\sqrt{3}}{2}\hat{\boldsymbol{e}}_2. \tag{1c}$$

(a) Find an expression for $\left(\frac{d}{dt}(\hat{\boldsymbol{e}}_1 + 5\hat{\boldsymbol{e}}_2 - 2\hat{\boldsymbol{e}}_3)\right)_{\mathrm{I}}$ (where subscript I denotes the inertial frame). You can express the answer either in terms of $\hat{\boldsymbol{e}}_i$ or $\boldsymbol{e}_i$.

(b) If a particle's position vector is defined as $\boldsymbol{X} = t^3\boldsymbol{e}_2$ in the inertial frame, find its acceleration in the rotating frame (written in terms of the unit vectors of the rotating frame).

Q2 The two-dimensional hydrostatic Boussinesq equations (where $\partial/\partial y = 0$ and $v = 0$) written in terms of the density variable are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x}, \tag{2a}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial z} - \frac{\rho g}{\bar{\rho}}, \tag{2b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2c}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = 0. \tag{2d}$$

(a) We can decompose the variables into a basic state (for the fluid at rest) and a perturbation

$$u = u'(x, z, t), \quad w = w'(x, z, t), \quad \rho = \tilde{\rho}(z) + \rho'(x, z, t), \quad p = \tilde{p}(z) + p'(x, z, t),$$

where $\tilde{(\ )}$ denotes the basic state and $(\ )'$ the perturbations to this state. Write (2) in terms of p', u', w' and ρ', and neglect the products of perturbation (primed) quantities. Assume that the basic state is at hydrostatic balance:

$$\frac{d\tilde{p}}{dz} = -\tilde{\rho}g.$$

(b) Write the equations you derive in part (a) in terms of buoyancy variable

$$b = -\frac{\rho'}{\bar{\rho}}g,$$

and buoyancy frequency

$$N^2 = -\frac{g}{\bar{\rho}}\frac{d\tilde{\rho}}{dz}.$$

Q3 Consider the magnetic field $\boldsymbol{B} = B_0 \sin(\alpha z)\boldsymbol{e}_x + B_0 \cos(\alpha z)\boldsymbol{e}_y$ where α and B_0 are constants.

(a) Calculate the current density.

(b) Is $\boldsymbol{B}$ a force-free field? Justify your answer.

(c) Calculate the magnetic pressure and tension forces associated with $\boldsymbol{B}$.

(d) Find a vector potential, $\boldsymbol{A}$, for $\boldsymbol{B}$.

Q4 (a) By considering $\boldsymbol{B}$ and $\boldsymbol{u}$ to be the sum of mean and fluctuating parts

$$\boldsymbol{B} = \langle \boldsymbol{B} \rangle + \boldsymbol{B}', \quad \boldsymbol{u} = \langle \boldsymbol{u} \rangle + \boldsymbol{u}',$$

such that $\langle \boldsymbol{B}' \rangle = \langle \boldsymbol{u}' \rangle = 0$, show that the mean field induction equation can be written as

$$\frac{\partial \langle \boldsymbol{B} \rangle}{\partial t} = \nabla \times (\langle \boldsymbol{u} \rangle \times \langle \boldsymbol{B} \rangle) + \nabla \times \langle \boldsymbol{u}' \times \boldsymbol{B}' \rangle + \eta \nabla^2 \langle \boldsymbol{B} \rangle.$$

(b) Hence or otherwise show that the induction equation for the fluctuating magnetic field can be written as

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \nabla \times (\langle \boldsymbol{u} \rangle \times \boldsymbol{B}') + \nabla \times (\boldsymbol{u}' \times \langle \boldsymbol{B} \rangle) + \nabla \times \mathcal{G} + \eta \nabla^2 \boldsymbol{B}',$$

where $\mathcal{G} = \boldsymbol{u}' \times \boldsymbol{B}' - \langle \boldsymbol{u}' \times \boldsymbol{B}' \rangle$.

(c) Consider the case where there is no mean flow, i.e., $\langle \boldsymbol{u} \rangle = 0$. Taking $\mathcal{B}$, $\mathcal{L}$, $\mathcal{T}$ and $\mathcal{U}$ to be characteristic scales of the fluctuating magnetic field, length, time and fluctuating velocity, respectively, show that, if $\mathcal{U} = \mathcal{L}/\mathcal{T}$,

$$|\nabla \times \mathcal{G}| \sim \left| \frac{\partial \boldsymbol{B}'}{\partial t} \right|.$$

Find also the characteristic scale of $\eta |\nabla^2 \boldsymbol{B}'|$ in terms of $Rm = \dfrac{\mathcal{U}\mathcal{L}}{\eta}$.

(d) For what values of Rm is it reasonable to neglect $\dfrac{\partial \boldsymbol{B}'}{\partial t}$ and $\nabla \times \mathcal{G}$ (in comparison with $\eta \nabla^2 \boldsymbol{B}'$) from the fluctuating induction equation? Comment on if this is a realistic assumption for astrophysical fluids.

SECTION B

Q5 We consider the Shallow Water equations (with the flat bottom) written in terms of the height variation from the mean $\xi = h - H$ (H being the mean height and constant) and the velocity vector $\boldsymbol{v} = (u, v)$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla_{\mathrm{H}})\boldsymbol{v} + f\boldsymbol{e}_z \times \boldsymbol{v} = -g\ \nabla_{\mathrm{H}}\xi \tag{3a}$$

$$\frac{\partial \xi}{\partial t} + H\ \nabla_{\mathrm{H}} \cdot \boldsymbol{v} + \nabla_{\mathrm{H}} \cdot (\xi \boldsymbol{v}) = 0, \tag{3b}$$

$$h(x, y, t) = \eta(x, y, t) = \xi(x, y, t) + H. \tag{3c}$$

where $f = (f_0 + \beta y)$ is the Coriolis parameter in the β-plane approximation and $\boldsymbol{e}_z$ the unit vector in the z direction. We consider the following scaling to non-dimensionalise this system

$$(x, y) = \mathcal{L}(x^*, y^*), \quad (u, v) = \mathcal{U}(u^*, v^*), \quad t = \mathcal{T}t^*, \quad \xi = \frac{f_0 \mathcal{L}\,\mathcal{U}}{g}\xi^*. \tag{4}$$

(a) Find the appropriate timescale $\mathcal{T}$ to derive the quasi-geostrophic (QG) dynamics for this system. You can do this by assuming $(\boldsymbol{v} \cdot \nabla_{\mathrm{H}})\boldsymbol{v}$ and $\dfrac{\partial \boldsymbol{v}}{\partial t}$ have similar orders.

(b) Non-dimensionalise (3) using the scaling given in (4) and part (a). Write your answer in terms of the Rossby number and the ratio of the deformation radius over the length scale:

$$Ro = \epsilon = \frac{\mathcal{U}}{f_0 \mathcal{L}}, \quad \frac{\mathcal{L}_d}{\mathcal{L}} = \frac{\sqrt{gH}}{f_0 \mathcal{L}}.$$

You should also assume that the beta effect is an order (ϵ) smaller than f_0:

$$\frac{f}{f_0} = \left(\frac{f_0 + \beta y^* \mathcal{L}}{f_0}\right) = \left(1 + \frac{\beta \mathcal{L}}{f_0} y^*\right) = (1 + \epsilon\ \beta^* y^*)\,.$$

where $\beta^* = \beta\mathcal{L}/(f_0 \epsilon)$ is $O(1)$ due to the assumption made.

(c) Expand the variables in terms of ϵ as below

$$\boldsymbol{v}^* = \boldsymbol{v}_0^* + \epsilon \boldsymbol{v}_1^*, \quad \xi^* = \xi_0^* + \epsilon \xi_1^*.$$

Then assuming $Ro = \epsilon \to 0$ and $\mathcal{L} \sim \mathcal{L}_d$, derive the leading-order equations.

(d) We define the dimensionless QG PV for this system as

$$q^* = \frac{\partial v_0^*}{\partial x^*} - \frac{\partial u_0^*}{\partial y^*} + \beta^* y^* - \left(\frac{\mathcal{L}}{\mathcal{L}_d}\right)^2 \xi_0^*.$$

Show that at the limit of $Ro = \epsilon \to 0$ and $\mathcal{L} \sim \mathcal{L}_d$ the dimensionless QG PV is materially conserved, i.e.

$$\frac{Dq^*}{Dt} = \frac{\partial q^*}{\partial t^*} + u_0^* \frac{\partial q^*}{\partial x^*} + v_0^* \frac{\partial q^*}{\partial y^*} = 0.$$

Q6 The Shallow Water equations (with the flat bottom and in an inertial frame) written in terms of the height variation from the mean ξ and the velocity vector $\boldsymbol{v} = (u, v)$ are

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla_{\mathrm{H}})\boldsymbol{v} = -g \ \nabla_{\mathrm{H}}\xi, \tag{5a}$$

$$\frac{\partial \xi}{\partial t} + H \ \nabla_{\mathrm{H}} \cdot \boldsymbol{v} + \nabla_{\mathrm{H}} \cdot (\xi \boldsymbol{v}) = 0. \tag{5b}$$

We consider the dynamics to consist of a background flow and waves

$$u = U + u', \quad v = v', \quad \xi = \xi', \tag{6}$$

where $U = \sin(\epsilon y)$ is the velocity of the background flow and u', v' and ξ' are the velocity and height variation of the waves. We assume ϵ to be a small parameter meaning that the background flow slowly varies with y (i.e. $dU/dy = O(\epsilon)$).

(a) Substitute (6) into (5) and linearise for the wave terms by neglecting the products of two wave terms. Also neglect the terms that are $O(\epsilon)$.

(b) Find the dispersion relation for the linearised equations (that you derive in part (a)) by assuming the following wave ansatz

$$u' = \tilde{u} \ e^{i(k_x x + k_y y - \omega t)}, \quad v' = \tilde{v} \ e^{i(k_x x + k_y y - \omega t)}, \quad \xi' = \tilde{\xi} \ e^{i(k_x x + k_y y - \omega t)}.$$

(c) Find the group velocity for each set of waves that you find in part (b).

Q7 Consider a conducting fluid flow down a channel whose boundaries are the planes $y = -d$ and $y = d$. The flow is given by $\boldsymbol{u} = u(y)\boldsymbol{e}_x$, where $u(y) = U \sin\left(\frac{\pi y}{d}\right)$ with U constant. At time $t = 0$, a uniform magnetic field $B_0 \boldsymbol{e}_y$ is applied. The flow then distorts the field so that at later times $\boldsymbol{B} = b(y,t)\boldsymbol{e}_x + B_0 \boldsymbol{e}_y$.

(a) Show that in this case the x-component of the induction equation becomes

$$\frac{\partial b}{\partial t} = B_0 \frac{du}{dy} + \eta \frac{\partial^2 b}{\partial y^2}.$$

(b) Show that for small t, b can be approximated by

$$b = \frac{B_0 U \pi}{d} \cos\left(\frac{\pi y}{d}\right) t.$$

(c) Given $b = 0$ at $y = \pm d$, find an approximate solution for b at large t.

(d) Sketch the magnetic field lines for the approximate solutions at both small and large t for $B_0 > 0$.

Q8 The following set of non-dimensional equations describes incompressible convection in the presence of a magnetic field in a layer of fluid lying between two horizontal planes at $z = 0$ and $z = 1$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + RaPrT\boldsymbol{e}_z + \zeta QPr(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + Pr\nabla^2 \boldsymbol{u},$$
$$\nabla \cdot \boldsymbol{u} = 0,$$
$$\frac{\partial T}{\partial t} + (\boldsymbol{u} \cdot \nabla)T = \nabla^2 T,$$
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \zeta \nabla^2 \boldsymbol{B},$$

where the relevant non-dimensional parameters are given by

$$Ra = \frac{\alpha \Delta T g d^3}{\nu \kappa}, \quad Pr = \frac{\nu}{\kappa}, \quad Q = \frac{B_0^2 d^2}{\mu_0 \bar{\rho} \nu \eta}, \quad \zeta = \frac{\eta}{\kappa}.$$

(a) Show that $\boldsymbol{u}_c = 0$, $T_c = 1 - z$, $p_c = \bar{p} + RaPr(z - z^2/2)$, $\boldsymbol{B}_c = \boldsymbol{e}_x$ is a basic state solution for this system, where $\bar{p}$ is a constant taken to be the value of p_c at the bottom of the layer.

(b) You are given that the equations governing the linear perturbations about the basic state in (a) can be written as

$$\frac{\partial \nabla^2 w'}{\partial t} = RaPr\nabla_H^2 \theta + \zeta QPr \frac{\partial \nabla^2 B_z'}{\partial x} + Pr\nabla^4 w',$$
$$\nabla \cdot \boldsymbol{u'} = 0,$$
$$\frac{\partial \theta}{\partial t} - w' = \nabla^2 \theta,$$
$$\frac{\partial B_z'}{\partial t} = \frac{\partial w'}{\partial x} + \zeta \nabla^2 B_z',$$

where w', θ and B_z' are the perturbations to the vertical velocity, temperature and vertical magnetic field, respectively.

By seeking normal mode solutions of the form $w' = \tilde{w}(z) f(x, y) \exp(st)$ (and similar for θ and B_z'), show that

$$\nabla_H^2 f = -k_h^2 f,$$

for some constant k_h.

Taking $f(x, y) = \exp(ik_x x + ik_y y)$, show that the normal mode equations can be written as

$$(D^2 - k_h^2)\left[s - Pr(D^2 - k_h^2)\right]\tilde{w} = -RaPrk_h^2\tilde{\theta} + \zeta QPrik_x(D^2 - k_h^2)\tilde{B}_z,$$
$$\left[s - (D^2 - k_h^2)\right]\tilde{\theta} = \tilde{w},$$
$$\left[s - \zeta(D^2 - k_h^2)\right]\tilde{B}_z = ik_x\tilde{w},$$

where $k_h^2 = k_x^2 + k_y^2$ and $D^2 \equiv \dfrac{\mathrm{d}^2}{\mathrm{d}z^2}$.

(c) Assume that the boundaries are impenetrable, free-slip and held at fixed temperature. Assume also that there is no vertical component of the magnetic field at the boundaries. Write down boundary conditions that $\tilde{w}$, $\tilde{\theta}$ and $\tilde{B}_z$ must satisfy and briefly justify your conditions.

Hint: the system requires 8 boundary conditions.

(d) Suggest suitable trigonometric forms for $\tilde{w}, \tilde{\theta}$ and $\tilde{B}_z$ and use these to find the critical value of Ra for the onset of convection as direct (non-oscillatory) modes.

What happens to the critical Rayleigh number if we consider disturbances aligned with the basic state magnetic field?