Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2024	MATH44320-WE02

Title: Advanced Probability Theory V: Paper 2

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2024

SECTION A

Q1 (a) State and prove the Second Moment Method for a non-negative integer-valued random variable.

(b) Suppose $c \in (0,1)$ and $\alpha > 0$ are constants, and for each $n \geq 1$ the random variables $X_1^{(n)}, X_2^{(n)}, \ldots$ are independent and identically distributed, with $X_1^{(n)} \sim \text{Bern}(cn^{-\alpha})$. Define $Y_n := \sum_{i=1}^{n} X_i^{(n)}$. Prove that, if $0 < \alpha < 1$, then $\mathbb{P}(Y_n = 0)$ converges as $n \to \infty$ and determine its limit.

(c) Describe the behaviour of $\mathbb{P}(Y_n = 0)$ as $n \to \infty$ for all other values of $\alpha > 0$.

Your answers should be supported by appropriate calculations and explanations.

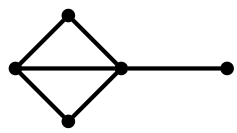
Q2 (a) Let T_d be the infinite regular tree in which every vertex has degree $d \geq 2$. Define the critical probability p_{cr} for Bernoulli **site** percolation on T_d, and prove that $p_{\mathsf{cr}} = 1/(d-1)$.

(b) Suppose T is an arbitrary infinite tree, with every vertex having degree in the range $[d_1, d_2]$ for integers d_1, d_2 satisfying $2 \leq d_1 \leq d_2$. Prove upper and lower bounds on the critical probability for Bernoulli site percolation on T.

Your answers should be supported by appropriate calculations and explanations.

SECTION B

Q3 Let H be the graph shown below, and let $G_{n,p}$ be a binomial random graph on n vertices, with edge probability $p \in [0,1]$.



(a) Using a coupling argument, or otherwise, prove that for all fixed n, the probability $\mathbb{P}(G_{n,p} \text{ contains a copy of } H)$ is non-decreasing in p.

(b) Define $X_{n,p}$ to be the number of copies of H in $G_{n,p}$. Calculate $\mathbb{E}X_{n,p}$ as a function of n and p, and show that as $n \to \infty$,

$$\mathbb{E}X_{n,p} \to \begin{cases} 0 & \text{if } pn^{\alpha} \to 0, \\ \infty & \text{if } pn^{\alpha} \to \infty, \end{cases} \qquad (\dagger)$$

where α is a constant that should be determined.

(c) State the definition of a threshold for containing a copy of H in the random graph $G_{n,p}$. For the value of α satisfying ($\dagger$) in part (b) above, explain whether $p^*(n) = n^{-\alpha}$ is a threshold for containing a copy of H. If $p^*(n)$ is not a threshold, find a function $p(n)$ for which $p(n)/p^*(n) \to \infty$ and $\mathbb{P}(G_{n,p(n)} \text{ contains a copy of } H) \to 0$ as $n \to \infty$.

Your answers should be supported by appropriate calculations and explanations.

Q4 (a) Suppose $G = (V, E)$ is an infinite, locally finite, connected graph. Describe the Bernoulli bond percolation model on G. Your answer should include a definition of the percolation probability $\theta_x^G(p)$ for a fixed vertex $x \in V$ and parameter $p \in [0, 1]$ and a definition of the critical probability $p_{\mathsf{cr}}(G)$.

(b) Let $\mathbb{L}^d$ be the d-dimensional integer lattice. Explain why the critical probability $p_{\mathsf{cr}}(\mathbb{L}^d)$ is non-increasing in d.

(c) Prove that $p_{\mathsf{cr}}(\mathbb{L}^d) < 1$ for all $d \geq 2$, by finding an explicit value of $p < 1$ with $p_{\mathsf{cr}}(\mathbb{L}^2) \leq p$.

Your answers should be supported by appropriate calculations and explanations.