Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH1031-WE01

Title: Discrete Mathematics

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

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Q1 (a) Consider the letters COMBINATION.

(i) How many arrangements of these letters are there?

(ii) How many arrangements contain the word NATION as a continuous string?

(iii) How many arrangements contain neither of the words NATION, BIN?

(b) Prove by induction that for any integer $n \geq 2$

$$1 \cdot 2^2 + 2 \cdot 3^2 + \ldots + (n-1)n^2 = \frac{n(n^2-1)(3n+2)}{12}.$$

Q2 (a) Solve the recurrence relation

$$a_n = 3a_{n-1} + 4a_{n-2} + 6n - 5$$

with initial condition $a_0 = -2$ and $a_1 = 4$.

(b) (i) Find the generating function for the sequence $(a_n, n \in \{0, 1, 2, \ldots\})$ where

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even } (0, 2, 4, 6, \ldots) \\ -1 & \text{if } n \text{ is odd } (1, 3, 5, 7, \ldots). \end{cases}$$

(ii) Find the generating function for the sequence $(b_n, n \in \{0, 1, 2, \ldots\})$ where

$$b_n = \begin{cases} n & \text{if } n \text{ is even } (0, 2, 4, 6, \ldots) \\ -n & \text{if } n \text{ is odd } (1, 3, 5, 7, \ldots). \end{cases}$$

(iii) Evaluate $\sum_{n=0}^{\infty} (-1)^n n 3^{-n}$.

Q3 (a) How many solutions $(x_1, x_2, x_3, x_4, x_5)$ are there to the equation

$$x_1 + x_2 - x_3 - x_4 + x_5 = 1,$$

where each x_i is an integer and

$$x_1 \leq 3, x_2 \leq 3, x_3 \geq -1, x_4 \geq 2, x_5 \leq 2?$$

(b) Jenny is choosing n of her old toys to send to a jumble sale. She can choose from (identical) dinosaur figures, (identical) building blocks, and colouring pens. The pens come in any of five colours, and are otherwise identical. She will choose at least three dinosaurs, at least two building blocks, and between 2 and 7 pens of each colour. Let d_n denote the number of ways in which Jenny can select n toys.

(i) Write down a generating function for d_n and express it as compactly as possible.

(ii) Use your generating function to find d_{25}.

Q4 The n-cube Q_n is the simple graph with vertex set given by binary strings of length n and edge set given by the pair of strings whose entries differ only at one position, i.e., $Q_n = (V_n, E_n)$, with $V_n := \{x : \{1, 2, \ldots, n\} \to \{0, 1\}\}$ and $E_n := \{\{x, y\} : x, y \in V_n, \sum_{i=1}^{n} |x(i) - y(i)| = 1\}$.

(a) How many edges are there in Q_n?

(b) Let G be a simple graph with n vertices. Prove that if the degree of each vertex is at least $n/2$, then G is connected.

(c) For which values of $n \geq 2$ is Q_n Hamiltonian? Justify your answer.

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