Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH1051-WE01

Title:
Analysis I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

Q1 Let $n \geq 2$ be a natural number.

(a) Show that
$$\sum_{m=2}^{n} m \cdot m! = (n+1)! - 2.$$

(b) Show that
$$\left(1 + \frac{1}{n^2 - 1}\right)^n > 1 + \frac{1}{n}.$$

[Hint: Use an inequality from the lectures.]

Q2 (a) Let $a, b \in \mathbb{R}$ with $a \neq b$. Show that
$$X = \{x \in \mathbb{R} \mid (x-a)(x-b) < 0\}$$
is non-empty and bounded, and determine $\sup(X)$. Also, decide whether X has a maximum. Justify your answer.

(b) Let $M \subset \mathbb{R} \setminus \{0\}$ be a non-empty set with $\inf(M) > 0$. Show that
$$N = \left\{x \in \mathbb{R} \;\middle|\; \frac{1}{x} \in M\right\}$$
is bounded, and show that $\sup(N) = \frac{1}{\inf(M)}$.

Q3 Calculate the limits of the following sequences, or show that the limit does not exist. State any results that you use.

(a) $x_n = \dfrac{\sqrt[n]{n} + \log n}{2^n}$,

(b) $x_n = \sqrt{9n^2 + 2n + 1} - \sqrt{9n^2 - 2n - 1}$,

(c) $x_n = \left(1 + \dfrac{1}{n+3}\right)^{n+1}$.

Q4 (a) For $n \in \mathbb{N}$ let
$$x_n = \frac{n + (-1)^n(2n+1)}{n}.$$
Determine $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

(b) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence and assume that
$$|x_{n+1} - x_n| < \frac{1}{2} \cdot |x_n - x_{n-1}|$$
for all $n \geq 2$. Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Q5 Determine whether the following series are convergent. If they are convergent, also determine whether they are absolutely convergent. State any results used.

(a) $\sum_{n=1}^{\infty} 10^{-n!}$,

(b) $\sum_{n=2}^{\infty} (-1)^n \frac{3}{\log n}$,

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2 - n + 1}$.

Q6 (a) Give the (ε, δ)-definition of continuity of a function $f : X \longrightarrow \mathbb{R}$ at a point $x = c \in X$, where X is an open subset of $\mathbb{R}$.

(b) Let $g : (1, \infty) \longrightarrow \mathbb{R}$ be function satisfying

$$|g(x) - g(y)| \leq M|\sqrt{x} - \sqrt{y}|$$

for all $x, y \in (1, \infty)$. Show, using the definition of continuity in (a), that g is continuous at any point $c > 1$.

(c) Give the negation of the statement (a); that is, the precise logical statement that the function f is not continuous at $x = c$. Quantifiers would be allowed.

Q7 (a) State the mean value theorem.

As a consequence derive the following fact: Let $f : I \to \mathbb{R}$ be a differentiable function on an interval I with $f'(x) \geq 0$ for all $x \in I$. Then f is monotonically increasing.

(b) For $r \neq 0$ any non-zero real number and $x > -1$, let $f(x) = (1+x)^r$. Compute what Taylor's Theorem (with Lagrange remainder) for $n = 1$ and $c = 0$ states for this function.

As a consequence derive the Bernoulli inequality for *real* exponent $r > 1$. That is,

$$(1 + x)^r \geq 1 + rx$$

for all $x > -1$.

Q8 (a) Find all $x \in \mathbb{R}$ such that $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^2\, 4^k}$ converges.

(b) Show that $f(x) := \sum_{k=1}^{\infty} \frac{k \cos(kx)}{3^k}$ is a continuous function on $\mathbb{R}$ and explicitly evaluate $\int_0^{\pi/2} f(x) dx$ as a (rational) number. Justify your reasoning.

Q9 Define a function on $I = [0, \infty)$ by $f(x) := \displaystyle\int_x^\infty \frac{1}{\sqrt{t^3+1}} dt$.

Throughout, do NOT try to evaluate the integral in any way.

(a) Show that the improper integral converges for all $x \in I$.

(b) Set $\alpha = f(0)$. Show that f is differentiable on $(0, \infty)$ and is a strictly monotone decreasing function mapping $[0, \infty)$ bijectively to $(0, \alpha]$.

(c) Show that the inverse function $g(u)$ of $f(x)$ exists and is differentiable on $(0, \alpha)$ satisfying $[g'(u)]^2 = g(u)^3 + 1$.

Q10 (a) Show that the sequence of functions $f_n(x) = \frac{1+x^n}{1-x^2}$ converges uniformly to $f(x) = \frac{1}{1-x^2}$ on compact subintervals $[-r, r]$ of $(-1, 1)$ with $0 < r < 1$.

(b) Give an example of a sequence of continuous functions $f_n(x)$ on $[0, 1]$ such that the pointwise limit $f(x)$ is *not* continuous. Conclude that the convergence $f_n \to f$ cannot be uniform.