Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH1061-WE01

Title:
Calculus I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

Q1 (a) Calculate

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}.$$

(b) Use the limit definition of the derivative to calculate the derivative of $f(x) = \sqrt{x}$.

Q2 Let

$$f(x) = \begin{cases} 2 - x^2 + |2x - x^2| & \text{for} \quad |x| \leq 3, \\ x & \text{for} \quad 3 < |x| \leq 5. \end{cases}$$

Find the global extremal values of $f(x)$ on the interval [-5,5].

Q3 Let D be the intersection of the disk of radius 2 centered on the origin and the upper half-plane with $y \geq 0$. Calculate

$$\iint_D x(x+y)\ dA.$$

Q4 (a) Find the general solution $y(x)$ to the ordinary differential equation

$$y'' + y = 2e^{-x}.$$

(b) Find the general solution $y(x)$ to the ordinary differential equation

$$y'' + y = \frac{1}{\sin(x)}.$$

Q5 (a) Use integration by parts to calculate

$$I_n = \int_{-\pi}^{\pi} \cosh(x) \cos(nx)\ dx$$

for each integer $n \geq 0$.

(b) By evaluating the Fourier series of $f(x) = \cosh(x)$ at $x = 0$, show that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(n^2+1)} = \frac{\pi}{2\sinh(\pi)}.$$

Q6 (a) A curve C in the x, y plane is given parametrically as $(x, y) = (t^3 - t, t^2 + t)$. Given that at the point $(x, y) = (0, 2)$, the function $f(x, y)$ is such that $f_x = 2$, $f_y = 3$, use the Chain Rule to calculate $\frac{df}{dt}$ when $t = 1$.

(b) Find and classify the stationary points of the function

$$f(x, y) = 2\ln(x) + x^2 + xy + \frac{y^2}{2} + x + 4y.$$

Q7 (a) A two-dimensional surface is given by the equation

$$f(x, y, z) \equiv x^3 + 3xy + 2yz + z^2 = -2.$$

Find the equation of the tangent plane to this surface at the point $(1, -1, 2)$.

(b) Find the Taylor expansion of the function

$$g(x, y) = y e^{\cos\left(\frac{x}{2}\right)}$$

about the point $(\pi, 1)$ to quadratic order.

Q8 A function $y(x)$ satisfies the differential equation

$$(x^2 + 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \lambda y$$

where λ is a constant.

(a) Explain briefly why you would expect that it is possible to write the solution in the form of a power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

(b) Find a recurrence relation satisfied by the coefficients a_n in the power series for $y(x)$.

(c) Use the recurrence relation from part (b) to find the value of λ for which the differential equation admits a solution in the form of a fifth order polynomial,

$$y(x) = x + \alpha x^3 + \beta x^5.$$

Here α and $\beta \neq 0$ are constants whose values you should determine to find the solution $y(x)$.

Q9 A function $u(x,t)$ satisfies the partial differential equation

$$u_t + u = 4u_{xx}$$

and the boundary conditions $u(0,t) = u(\pi,t) = 0$.

(a) For each integer $n \geq 0$, find the value of μ_n for which

$$u_n(x,t) = \sin(nx)e^{\mu_n t}$$

is a solution both to the partial differential equation and the boundary conditions given above.

(b) Find the solution to the partial differential equation satisfying the boundary conditions as a linear combination of the solutions from part (a) $u(x,t) = \sum_{n=1}^{\infty} A_n u_n(x,t)$ by determining the coefficients A_n if at time $t = 0$

$$u(x,0) = \begin{cases} \sin(\alpha x) & \text{for } 0 < x < \frac{\pi}{\alpha} \\ 0 & \text{for } \frac{\pi}{\alpha} < x < \pi \end{cases}$$

where α is a constant greater than one and not an integer. It may help to use the trigonometric identity

$$\sin(\alpha x)\sin(\beta x) = \frac{1}{2}\cos((\alpha-\beta)x) - \frac{1}{2}\cos((\alpha+\beta)x).$$

Q10 (a) Write down the definition of the Fourier transform $\tilde{f}(p)$ of the function $f(x)$, and also the equation which gives $f(x)$ in terms of its Fourier transform $\tilde{f}(p)$.

(b) You are given that the Fourier transform of $f(x) = \frac{1}{\cosh(x)}$ is

$$\tilde{f}(p) = \frac{\pi}{\cosh\left(\frac{\pi p}{2}\right)}.$$

By using the Scaling theorem, find a function $g(x)$ such that $\tilde{g}(p) = \Lambda g(p)$ where you should explicitly find the constant Λ.

(c) By considering the Fourier transform of $\frac{d^2 f}{dx^2}$ where $f(x)$ is the function given in part (b), find the Fourier transform of $(\cosh(x))^{-3}$.