Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH1091-WE01

Title:
Linear Algebra I (Maths Hons)

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

Q1 Let $L, \Pi \subset \mathbb{R}^3$ be the line and the plane defined by

$$L = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

and

$$\Pi = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + 3y + 2z = 1 \right\}.$$

Find all elements of the set $L \cap \Pi$.

Q2 Consider the system of linear equations

$$\begin{aligned} x + y + z &= 0, \\ 2x + y - z &= 0, \\ tx + 2y + 4z &= 0. \end{aligned}$$

For each value of $t \in \mathbb{R}$, determine the solution set of this linear system.

Q3 Determine whether or not the following matrix has an inverse. Give the inverse matrix if it exists.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Q4 Consider the linear map

$$\phi\colon \mathbb{R}[x]_4 \longrightarrow \mathbb{R}^3$$

given by

$$\phi\colon f \longmapsto \begin{pmatrix} f(0) \\ f(1) \\ f'(0) \end{pmatrix},$$

(where we write f' for the derivative of f).

(a) Show that ϕ is surjective and hence determine the rank and nullity of ϕ.

(b) Find a basis for $\ker(\phi)$.

Q5 Let

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

and consider the map

$$\phi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

given by

$$\phi\colon \boldsymbol{v} \longmapsto (\boldsymbol{v} \times \boldsymbol{a}) - \boldsymbol{v}$$

where we write $\times$ for the usual cross product.

(a) Show that ϕ is a linear map.

(b) Write down the matrix of ϕ with respect to the standard basis of $\mathbb{R}^3$.

(c) Hence or otherwise show that ϕ is an isomorphism.

Q6 Let A be the matrix

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}$$

Compute the characteristic polynomial of A, and use this to evaluate A^5. (Hint: use Cayley-Hamilton multiple times.)

Q7 Given

$$B = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

find a matrix P such that $P^{-1}BP$ is diagonal, and check your result by a direct calculation.

Q8 Let V be a vector space and let $T : V \to V$ be a linear mapping which satisfies $T^3 = 1$, where 1 denotes the identity mapping. Define a further linear mapping $S : V \to V$ by

$$S = \frac{1}{3}(1 + T + T^2)\,.$$

Show that $S^2 = S$, and that any vectors in the image of S are eigenvectors of T with eigenvalue 1.

Q9 Find all values of $a, b \in \mathbb{C}$ for which the sesquilinear form

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle = z_1\overline{w}_1 + az_2\overline{w}_2 + 2iz_1\overline{w}_2 + bz_2\overline{w}_1$$

on $\mathbb{C}^2$ is Hermitian, so that $\langle \boldsymbol{z}, \boldsymbol{w} \rangle = \overline{\langle \boldsymbol{w}, \boldsymbol{z} \rangle}$. For which of these values does $\langle\,,\rangle$ define a complex inner product?

For such values of a and b, find all vectors $\boldsymbol{u} \in \mathbb{C}^2$ orthogonal to

$$\boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;,$$

with respect to the complex inner product defined by $\langle\,,\rangle$.

Q10 Let G be the set of lower-triangular 3×3 matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$$

where a, b and c are even integers (or zero). Prove that G is a group under matrix multiplication. (You may assume that matrix multiplication is associative.)

END