Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH1561-WE01

Title:
Single Mathematics A

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

Q1 (a) Differentiate $x^{x\cos(x)}$ with respect to x.

(b) Compute the following two limits:

$$\lim_{x\to 1}\frac{\cos(\pi x/2)}{1-x^2} \qquad\qquad \lim_{x\to\infty} e^{-x}\sin(x^2)$$

Clearly state which theorem you have used, if any.

(c) State the definition of the derivative of a function $f(x)$ as a limit. By working out this limit find the derivative of

$$f(x)=\frac{x+1}{x-2}$$

Note: for part (c) do *not* use L'Hôpital's rule!

Q2 (a) Compute the following definite integral:

$$\int_0^{\pi/4}\cos^3(x)\sin^5(x)dx$$

(b) Compute the following indefinite integral:

$$\int\frac{2x}{(x+1)^2(x^2+2x+2)}dx$$

Hint: $\int 1/(1+u^2)du=\arctan(u)+c$

(c) Show that, for $z=x+iy$,

$$|\sin(z)|^2=a\cos(2x)+b\cosh(2y)$$

where a and b are numbers which you should determine.

Q3 (a) Find the modulus and argument of

$$\exp\left(\frac{4+i}{1+i}\right)$$

(b) State de Moivre's theorem.

(c) Express $\cos(4\theta)$ as a polynomial in $\cos(\theta)$.

(d) Find all complex solutions z to the equation

$$z^4-2z^2+4=0$$

Give all your answers in polar form.

Please turn over!

Q4 (a) Determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{n!(n+4)!}{(2n)!}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n - 2e^{-n}}$$

converge or not. If yes, decide whether the series is also absolutely convergent.

(b) Determine the interval of convergence for the power series

$$\sum_{n=0}^{\infty} (2^n + 3^n)(x-1)^n.$$

Q5 (a) Let $f(x) = xe^{\sin(x)}$.

(i) Find the third-order Taylor polynomial $p_3(x)$ of $f(x)$ about $x = 0$.

(ii) Compute the second-order Taylor polynomial approximation $p_2(1/4)$ of $f(1/4)$ and use the Lagrange form of the remainder and $|\sin(x)| \leq |x|$ to show

$$\left| f\left(\frac{1}{4}\right) - p_2\left(\frac{1}{4}\right) \right| \leq \frac{5}{6} \cdot \frac{1}{4^3} \cdot e^{1/4}.$$

(b) Consider the vectors

$$\begin{pmatrix} a \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 9 \\ 3 \\ 6 \end{pmatrix}.$$

(i) Find the value $a \in \mathbb{R}$ for which these two vectors are linearly dependent.

(ii) Compute the determinant of the matrix

$$M(a,b) = \begin{pmatrix} a & 9 & b \\ 1 & 3 & b^2 - 3b + 2 \\ 2 & 6 & 0 \end{pmatrix}.$$

Under which conditions of a and b is this determinant non-zero?

Q6 Consider the following inhomogeneous system of linear equations.

$$\begin{aligned} x + y + z &= c \\ x + (c+1)y + z &= 2c \\ x + y + (1+c)z &= 0 \end{aligned}$$

(a) For which values of $c \in \mathbb{R}$ does the system of linear equations have (i) no solutions, (ii) a unique solution, (iii) infinitely many solutions?

Find the solutions in cases (ii) and (iii) and, in case (iii), also say whether the solutions represent a line or a plane.

(b) Rewrite the above system of equations in the form

$$Av = b$$

with $v = \begin{pmatrix} x & y & z \end{pmatrix}^\top$ and compute A^{-1} in the case (ii).

Please turn over!

Q7 (a) Consider the matrix

$$A := \begin{pmatrix} 5 & -5 & 5 \\ 4 & -7 & 4 \\ 2 & -1 & 2 \end{pmatrix}.$$

Two of the eigenvalues of A are -5 and 5 with corresponding eigenvectors $\begin{pmatrix} 1 & 2 & 0 \end{pmatrix}^\top$ and $\begin{pmatrix} 2 & 1 & 1 \end{pmatrix}^\top$, respectively. (You do **NOT** need to show this!) Find the third eigenvalue of A and an invertible matrix M such that $M^{-1}AM$ is diagonal. It is sufficient to find M, you do **NOT** need to compute M^{-1} and to check whether $M^{-1}AM$ is diagonal.

(b) Decide about the validity of each of the following statements about quadratic matrices of the same size. Give a proof for the statements which are true, and provide a counterexample for the statements which are false:

(i) If B, C are symmetric matrices, then $B + C$ is also symmetric.

(ii) If B, C are orthogonal matrices, then $B + C$ is also orthogonal.

(iii) If B, C are orthogonal matrices, then BC is also orthogonal.

(iv) If B is both orthogonal and skew-symmetric, then $B^2 = -\mathrm{Id}$.