Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH1571-WE01

Title: Single Mathematics B

Time:	3 hours	
Additional Material provided:	Tables: Normal Distribution	
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

Q1 Four points A, B, C and D are given in $\mathbf{R}^3$ as

$$A = (1, 2, 0), \qquad B = (-1, 4, 2), \qquad C = (3, 0, 1), \qquad D = (-2, 3, -1)\,.$$

(a) Find the vector equation for the line ℓ_1 through the points A and B and the vector equation for the line ℓ_2 through the points C and D.

(b) Determine the shortest distance between ℓ_1 and ℓ_2.

(c) Give the magnitude of the two vectors $\overrightarrow{BA}$ and $\overrightarrow{AC}$ as well as the cosine of the angle between them.

(d) Compute the volume of the tetrahedron with vertices A, B, C and D.

Q2 (a) For an arbitrary real constant a, find a solution $y = y(x, a)$ to the following ordinary differential equation

$$\frac{dy}{dx} + 2xy = ax$$

with boundary condition

$$y(1) = 2\,.$$

(b) On a small island, there is a wolf population $W = W(t)$ and a sheep population $S = S(t)$. According to the biologists' records, who denote the growth rate of the sheep by a, the devouring impact of the wolves on the sheep population by b, the death rate of the wolves by c and the nourishing impact of the sheep on W by f, the following changes over time, with constants a, b, c and f, have emerged

$$\begin{aligned} \frac{dS}{dt} &= aS - bW\,, \\ \frac{dW}{dt} &= -cW + fS\,. \end{aligned}$$

(i) Deduce a second order differential equation for $S(t)$ from these.

(ii) Give the general solution of the second order equation in the previous part for the special case $b = f = \frac{a+c}{2}$.

(iii) If, moreover, $a < c$ (and still $b = f = \frac{a+c}{2}$), what happens to the sheep population over time?

Q3 (a) Let $h(x)$ be the 2π-periodic function given on the interval $(-\pi, \pi)$ by

$$h(x) = \begin{cases} -1 & \text{if } \; -\pi < x < -\frac{\pi}{2}\,, \\ 0 & \text{if } \; -\frac{\pi}{2} < x < \frac{\pi}{2}\,, \\ 1 & \text{if } \; \frac{\pi}{2} < x < \pi\,, \end{cases}$$

and where $h(\pi) = 0$.

Draw the function in the interval $(-2\pi, 2\pi)$ and give the Fourier expansion of $h(x)$ in terms of trigonometric expressions. Give the expansion explicitly up to the first six non-zero coefficients.

(b) Give the identity in Parseval's Theorem, carefully explaining the meaning of the terms involved.

(c) Assuming Parseval's Theorem holds for the function $h(x)$ above, deduce the value of

$$\sum_{n\geq 1} \frac{1}{n^2}\big(\cos(n\pi) - \cos(n\frac{\pi}{2})\big)^2\,.$$

Q4 (a) If $f(x, y)$ is a function of x and y, where $x = u^2 + v^2$ and $y = u^2 - v^2$, show that

$$uvf_{uv} = (x^2 - y^2)(f_{xx} - f_{yy}),$$

where the subscripts indicate partial differentiation. (You may assume that $f_{xy} = f_{yx}$.)

(b) Find and classify the critical points of the function

$$f(x, y) = x^3 - 12xy - 3x + 9y^2\,.$$

(c) The temperature in a lake is given by

$$T(x, y, z) = 4x^2 + y^2 - z^2 + 7,$$

where the surface is at $z = 0$. A diver at the position $(1, -2, -4)$ begins to feel cold and wishes to warm up as quickly as possible. In which direction, written as a unit 3-vector, should the diver swim? What is the rate at which the water temperature increases in this direction?

Q5 (a) Determine all constant parameters a and b such that the differential

$$df = (2x + ay)\, dx + (bx + 6y)\, dy\,.$$

is exact and, for these values of a and b, determine a function $f(x, y)$ whose total differential is df.

(b) Consider the vector field

$$\boldsymbol{V} = (V_1, V_2, V_3) = (2x + ye^x \cos(z), e^x \cos(z), -ye^x \sin(z))\,.$$

Determine:

(i) $\nabla \cdot \boldsymbol{V}$

(ii) $\nabla \times \boldsymbol{V}$

(iii) $\nabla \times (z^3 \boldsymbol{V})$

(c) Let $\boldsymbol{W} = (W_1, W_2, W_3)$ be a vector field satisfying

$$\frac{\partial W_3}{\partial z} = 0\,, \quad \nabla \times \boldsymbol{W} = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{W} = g(z),$$

where $g(z)$ is a function depending only on z (i.e. it does not depend on x and y). Show that the x-component W_1 and y-component W_2 of $\boldsymbol{W}$ each satisfy the two-dimensional Laplace equation; that is, show that

$$\frac{\partial^2 W_1}{\partial x^2} + \frac{\partial^2 W_1}{\partial y^2} = 0 = \frac{\partial^2 W_2}{\partial x^2} + \frac{\partial^2 W_2}{\partial y^2}\,.$$

Q6 (a) Evaluate the integral

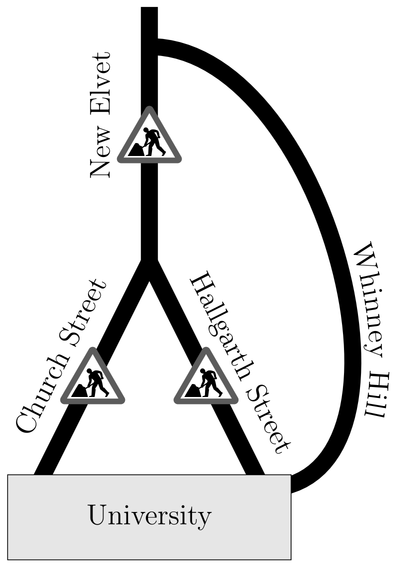
$$\iint_R (1 + 2x^3 y)\, dA$$

where R is the finite region bounded by the y-axis, the positive x-axis and the curve $y = 4 - x^2$.

(b) Use integration to find the volume of the finite region enclosed by the ellipsoid

$$4x^2 + 4y^2 + 9z^2 = 9.$$

Q7 (a) Durham County Council is planning roadworks on New Elvet, on Church Street and on Hallgarth Street, as illustrated in the diagram below. Each set of roadworks will be scheduled independently of the others (a different crew will carry out each set of roadworks) and, in each case, traffic will not be able to pass along the affected road for the duration of the work. If these roads cannot provide a route to the university, traffic will be diverted along Whinney Hill.



The respective probabilities that roadworks will take place on each road during the Single Mathematics B exam are

p_1 for New Elvet, p_2 for Church Street and p_3 for Hallgarth Street.

Assume that $p_1 p_2 p_3$ is the probability that during the Single Mathematics B exam roadworks will take place on all three roads simultaneously.

(i) Write the event that traffic will be diverted along Whinney Hill in terms of the events that roadworks take place on the other three roads.

(ii) Using part (i) or otherwise, show that the probability that traffic to the university will be diverted along Whinney Hill is $p_1 + p_2 p_3 - p_1 p_2 p_3$.

(iii) Find the (conditional) probability that traffic will be diverted along Whinney Hill, given that there will be roadworks on New Elvet.

(iv) Find the (conditional) probability that there will be roadworks on New Elvet, given that traffic will be diverted along Whinney Hill.

(b) A laboratory tests blood samples for a particular chronic disease. It is known that a proportion 0.8% of the total population has the disease. The laboratory tests 1000 independent samples a week, taken uniformly at random from the population.

(i) What type of probability distribution should the number of positive tests per week have, including the value of any relevant parameters?

(ii) Using an appropriate approximation, calculate the probability that the number of positive tests is less than or equal to 2.

(iii) Suppose the laboratory keeps a record of the number of positive test results per week for a whole year (i.e. 52 weeks of data). Approximate the probability that the average number of positive tests is less than 7.