

EXAMINATION PAPER

Examination Session: May/June

2025

Year:

Exam Code:

MATH1597-WE01

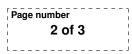
Title:

Probability I

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

All questions carry the same marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.	Instructions to Candidates:	barcodes.
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Revision:

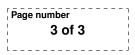


- **Q1** A gambler has in his pocket a fair coin, and a two-headed coin (a coin which shows heads on both sides). Assume that he is equally likely to choose either of the two coins.
 - (a) He selects one of the coins at random, and flips it. If the flip shows heads, find the probability that he chose the fair coin.
 - (b) Suppose that he flips the same coin a second time, and again it shows heads. Now find the probability that it is the fair coin.
 - (c) Suppose that he flips the same coin a third time, and this time it shows tails. Now find the probability that it is the fair coin.
- Q2 A biased coin is tossed n times. Let p denote the probability of heads in each toss, where $0 . Let <math>H_n$ and T_n denote the number of heads and tails obtained in n tosses, and let $Y_n = H_n T_n$ denote the difference between the number of heads and the number of tails.
 - (a) What is the probability mass function of H_n ?
 - (b) Show that $\mathbb{E}[H_n] = np$.
 - (c) Let $0 \le j \le n$ and assume that n+j is even. Find $\mathbb{P}(Y_n = j)$. [Hint: Can you say anything about $H_n + T_n$?].
 - (d) Find $\text{Cov}(H_n, Y_n)$. You may use the fact that $\text{Var}(H_n) = np(1-p)$. State clearly any property of covariance that you are using. [Hint: Can you say anything about $H_n + T_n$?].
- **Q3** Let X and Y be random variables, with joint probability mass function given by

$$f(x,y) = \begin{cases} c(xy+y^2), & 0 \le x \le 1, \ 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where c > 0 is a constant.

- (a) Find the value of c.
- (b) Find the marginal probability density functions of X and Y.
- (c) Find $\mathbb{P}(X < Y)$.
- (d) Find the $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
- Q4 Let Z be a random variable which has a standard normal distribution, that is, $Z \sim \mathcal{N}(0, 1)$.
 - (a) Find the moment generating function of Z. Show your working clearly, and justify each step.
 - (b) Find $\mathbb{E}[Z^3]$ and $\mathbb{E}[Z^4]$ using the moment generating function of Z. State clearly any property of the moment generating function that you use.
 - (c) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the moment generating function of X using the moment generating function of a standard normal random variable.
 - (d) Find $\mathbb{E}[X^3]$. [Hint: You can use the answer that you obtained for $\mathbb{E}[Z^3]$].



Q5 The lifespan of a lightbulb is distributed as $Exp(\lambda)$ for $\lambda > 0$, that is, if X denotes the lifespan of a lightbulb, then the probability density function of X is given by

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$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \ge 0\\ 0, & \text{elsewhere.} \end{cases}$$

A lightbulb is used until it fails, when it is replaced by a new one, the lifespan of which is independent and identically distributed as $Exp(\lambda)$. Let X_1, X_2, X_3, \ldots denote the sequence of lifespans of the lightbulbs, and let

$$S_n = X_1 + X_2 + \dots + X_n$$

denote the total lifespan of the first n lightbulbs.

In this question, you may assume that the expectation of the $Exp(\lambda)$ distribution is $\frac{1}{\lambda}$, and its variance is $\frac{1}{\lambda^2}$.

(a) Carefully state the Weak Law of Large Numbers for S_n as $n \to \infty$.

(b) Let n = 25 and $\lambda = \frac{1}{40}$. Using Chebyshev's inequality, show that

$$\mathbb{P}(700 \le S_{25} \le 1300) \ge \frac{5}{9}$$

(c) With the same n and λ as in part (b), use the Central Limit Theorem to find an approximate value of

$$\mathbb{P}(1050 \le S_{25} \le 1100).$$

You may consult the table below for the values of the cumulative distribution function of a standard normal random variable.

The table below gives the cumulative distribution function for a standard normal random variable.

z	0.25	0.35	0.50	0.65
$\Phi(z)$	0.5987	0.6554	0.6915	0.7422