Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH1597-WE01

Title:
Probability I

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

Q1 A gambler has in his pocket a fair coin, and a two-headed coin (a coin which shows heads on both sides). Assume that he is equally likely to choose either of the two coins.

(a) He selects one of the coins at random, and flips it. If the flip shows heads, find the probability that he chose the fair coin.

(b) Suppose that he flips the same coin a second time, and again it shows heads. Now find the probability that it is the fair coin.

(c) Suppose that he flips the same coin a third time, and this time it shows tails. Now find the probability that it is the fair coin.

Q2 A biased coin is tossed n times. Let p denote the probability of heads in each toss, where $0 < p < 1$. Let H_n and T_n denote the number of heads and tails obtained in n tosses, and let $Y_n = H_n - T_n$ denote the difference between the number of heads and the number of tails.

(a) What is the probability mass function of H_n?

(b) Show that $\mathbb{E}[H_n] = np$.

(c) Let $0 \leq j \leq n$ and assume that $n + j$ is even. Find $\mathbb{P}(Y_n = j)$. [Hint: Can you say anything about $H_n + T_n$?].

(d) Find $\mathrm{Cov}(H_n, Y_n)$. You may use the fact that $\mathrm{Var}(H_n) = np(1-p)$. State clearly any property of covariance that you are using. [Hint: Can you say anything about $H_n + T_n$?].

Q3 Let X and Y be random variables, with joint probability mass function given by

$$f(x,y) = \begin{cases} c(xy + y^2), & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $c > 0$ is a constant.

(a) Find the value of c.

(b) Find the marginal probability density functions of X and Y.

(c) Find $\mathbb{P}(X < Y)$.

(d) Find the $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Q4 Let Z be a random variable which has a standard normal distribution, that is, $Z \sim \mathcal{N}(0,1)$.

(a) Find the moment generating function of Z. Show your working clearly, and justify each step.

(b) Find $\mathbb{E}[Z^3]$ and $\mathbb{E}[Z^4]$ using the moment generating function of Z. State clearly any property of the moment generating function that you use.

(c) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the moment generating function of X using the moment generating function of a standard normal random variable.

(d) Find $\mathbb{E}[X^3]$. [Hint: You can use the answer that you obtained for $\mathbb{E}[Z^3]$].

Q5 The lifespan of a lightbulb is distributed as $Exp(\lambda)$ for $\lambda > 0$, that is, if X denotes the lifespan of a lightbulb, then the probability density function of X is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

A lightbulb is used until it fails, when it is replaced by a new one, the lifespan of which is independent and identically distributed as $Exp(\lambda)$. Let $X_1, X_2, X_3, \ldots$ denote the sequence of lifespans of the lightbulbs, and let

$$S_n = X_1 + X_2 + \cdots + X_n$$

denote the total lifespan of the first n lightbulbs.

In this question, you may assume that the expectation of the $Exp(\lambda)$ distribution is $\frac{1}{\lambda}$, and its variance is $\frac{1}{\lambda^2}$.

(a) Carefully state the Weak Law of Large Numbers for S_n as $n \to \infty$.

(b) Let $n = 25$ and $\lambda = \frac{1}{40}$. Using Chebyshev's inequality, show that

$$\mathbb{P}(700 \leq S_{25} \leq 1300) \geq \frac{5}{9}.$$

(c) With the same n and λ as in part (b), use the Central Limit Theorem to find an approximate value of

$$\mathbb{P}(1050 \leq S_{25} \leq 1100).$$

You may consult the table below for the values of the cumulative distribution function of a standard normal random variable.

The table below gives the cumulative distribution function for a standard normal random variable.

z	0.25	0.35	0.50	0.65
$\Phi(z)$	0.5987	0.6554	0.6915	0.7422