Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2011-WE01

Title:
Complex Analysis II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

SECTION A

Q1 Consider the function

$$f(z) = f(x+iy) = x^4 - 6x^2y^2 + y^4 + i\left(4x^3y - 4xy^3\right).$$

(a) At which points is f complex differentiable? Justify your answer.

(b) At which points is f holomorphic? Justify your answer.

Q2 (a) Use the definition of $\sin(z)$ and $\cos(z)$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

to show that for any $z \in \mathbb{C}$

$$2\sin(2z)\sin(z) = \cos(z) - \cos(3z).$$

(b) Using the principal branch of the logarithm calculate $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{\frac{\pi}{2}}$.

Q3 Let a, b, c be complex numbers, with $a \neq 0$, and let γ denote the contour $|z| = 3$ traversed anticlockwise. For $\omega \in B_3(0) = \{z \in \mathbb{C} : |z| < 3\}$ consider the function

$$g(\omega) = \frac{1}{2\pi i}\int_\gamma \frac{az^2 + bz + c}{z - \omega}\, dz \qquad (|\omega| < 3).$$

(a) Evaluate $g(\omega)$, stating clearly any results that you use.

(b) Hence, explain why g can have at most two fixed points in $B_3(0)$; that is, points ω with $|\omega| < 3$ such that $g(\omega) = \omega$.
Find these fixed points in the case that $a = b = 1$ and $c = -1$.

Q4 Suppose f is holomorphic on $\mathbb{C}$ and consider the function $g(z) = \exp(f(z))$.

(a) By differentiating g, or otherwise, show that if g were constant on $\mathbb{C}$ then f would also have to be constant on $\mathbb{C}$.

(b) Suppose there exists some real constant $M > 0$ such that $|\mathrm{Re}(f(z))| \leq M$ for all $z \in \mathbb{C}$. By proving that g is bounded on $\mathbb{C}$, show that f must be constant on $\mathbb{C}$. State any result that you use.

(c) Explain why this means that all bounded *harmonic* functions on $\mathbb{C}$ are constant.

SECTION B

Q5 (a) Prove that the general form of a Möbius transformation that takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to the right half plane $\mathbb{H}_R = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is

$$M(z) = -i\frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

(b) Show that if $M(z)$ is a Möbius transformation that takes the upper half plane $\mathbb{H}$ to the right half plane $\mathbb{H}_R$ and if $M(i) = 1$ then

$$M(-i) = -1.$$

(c) Let $M(z)$ be a Möbius transformation that takes the upper half plane $\mathbb{H}$ to the right half plane $\mathbb{H}_R$ and such that $M(i) = 1$ and $M(0) = 0$. Find the image of the unit disc $\mathbb{D}$ under $M(z)$.

Q6 *In this question you may use without proof the fact that that for $z = x + iy$*

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

Consider the series

$$\sum_{n=0}^{\infty} \frac{\cos^{2n}(z)}{n!} \tag{1}$$

(a) Show that for any $\alpha > 0$ the series converges uniformly in

$$V_\alpha = \{z \in \mathbb{C} \ : \ -\alpha < \mathrm{Im}(z) < \alpha\}.$$

(b) Show that for any $\beta > 0$ the sequence of functions

$$f_n(z) = \frac{\cos^{2n}(z)}{n!}$$

does not converge uniformly to $f(z) = 0$ on

$$D_\beta = \{z \in \mathbb{C} \ : \ -\beta < \mathrm{Re}(z) < \beta\}$$

by finding an appropriate purely imaginary sequence $\{z_n\}_{n\in\mathbb{N}}$, i.e. a sequence $\{z_n\}_{n\in\mathbb{N}} \subset i\mathbb{R}$.

Q7 *In this question you may use without proof the fact that the zeros of $g(z) = \sin(z)$ lie at precisely the points $z = \pi k$ for $k \in \mathbb{Z}$, and that $e - 1/e \geq 2$.*

Consider the function $f(z) = \sin(z) - 1/z$, defined for all $z \in \mathbb{C} \setminus \{0\}$. Let

$$L = \left\{ z \in \mathbb{C} : \mathrm{Re}(z) \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) \right\}$$

be an open vertical strip in $\mathbb{C} \setminus \{0\}$ and for any real $R \geq 1$ define the subset

$$L(R) = \{ z \in L : \mathrm{Im}(z) \in (-R, R) \} .$$

Denote by $\ell(R)$ the rectangular contour bounding $L(R)$, traversed anticlockwise.

(a) Show that on the vertical lines $\mathrm{Re}(z) = \pi \pm \pi/2$ we have

$$|\sin(z)| = \frac{1}{2}(e^{\mathrm{Im}(z)} + e^{-\mathrm{Im}(z)}) = \cosh(\mathrm{Im}(z)).$$

Similarly, show that on the horizontal lines $\mathrm{Im}(z) = \pm R$ we have

$$|\sin(z)| \geq \frac{1}{2}(e^{R} - e^{-R}).$$

Hence, conclude that $|\sin(z)| \geq 1$ for all z on the contour $\ell(R)$, when $R \geq 1$.

(b) Use Rouché's Theorem to show that for all $R \geq 1$ the function f has exactly one zero in $L(R)$. Explain why this means that f has exactly one zero in the vertical strip L.

(c) Assume that f has no zeros on the vertical lines $\mathrm{Re}(z) = \pi \pm \pi/2$, and consider the contour $\ell(\pi)$ corresponding to the case $R = \pi$. Using part **7(b)**, evaluate the integral

$$\int_{\ell(\pi)} \frac{f'(z)}{f(z)}\, dz \;=\; \int_{\ell(\pi)} \frac{z^2 \cos(z) + 1}{z(z \sin(z) - 1)}\, dz.$$

State any results that you use.

Q8 Consider the meromorphic function $f(z) = e^{-2iz}\,(1 + z^2)^{-1}$.

(a) For $R > 1$ let C_R be the semicircular contour in the lower half-plane defined by $C_R(\theta) = Re^{i\theta}$ for $\theta \in [-\pi, 0]$. Show that

$$\int_{C_R} f(z)\, dz \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.$$

(b) Let L_R be the straight line segment on the real axis that begins at the point $z = R$ and ends at the point $z = -R$. By integrating f around the 'D-shaped' contour given by $L_R \cup C_R$, evaluate

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{e^{-2ix}}{1 + x^2}\, dx.$$

(c) In answering part **8(b)**, why could we not integrate f around the usual D-shaped contour consisting of the line segment joining $z = -R$ to $z = R$ and the semicircular contour in the upper half-plane given by $C_R(\theta) = Re^{i\theta}$ for $\theta \in [0, \pi]$?