

EXAMINATION PAPER

Examination Session: May/June Year: 2025

Exam Code:

MATH2031-WE01-SP

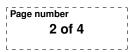
Title:

Analysis in Many Variables II (2023/24 Syllabus)

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Materials i erritted.		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions.	
	Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.	
	Write your answer in the white-covered answer booklet with barcodes.	
	Begin your answer to each question on a new page.	

Revision:



SECTION A

- **Q1** Let $f(x) = z \cos(rz)e_r + [r \cos(rz) + 1]e_z$ in cylindrical coordinates.
 - (a) By finding a suitable potential $g(\boldsymbol{x})$, show that \boldsymbol{f} is conservative in other words, may be written $\boldsymbol{f} = \boldsymbol{\nabla} g$.
 - (b) Naming any theorem that you use, compute the line integral of f along an arbitrary curve from $(r, \theta, z) = (0, 0, 0)$ to $(r, \theta, z) = (3, \pi, \pi/2)$.
- Q2 (a) Calculate the volume of the region $\{-1 \le z \le 1, r < 1 z^2\}$ in cylindrical coordinates.
 - (b) Use index notation to simplify as much as possible the expression $\nabla \cdot (\boldsymbol{x} \times \boldsymbol{a})$, where \boldsymbol{a} is a constant vector in \mathbb{R}^3 .
- **Q3** Solve the following equation for the generalised function g,

$$(x-1)(x^2 + x - 2)g(x) = 0,$$

i.e. find the generalised solution g(x) in terms of shifted δ_a distributions and possibly their derivatives. Justify the steps taken to arrive at the solution.

 $\mathbf{Q4}$ You are given the linear operator

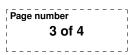
$$L = (1 - x^2) \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x),$$

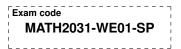
with the two real-valued functions $g \in \mathcal{C}^1((-1,1))$ and $h(x) \in \mathcal{C}^0((-1,1))$.

- (a) Calculate the formal adjoint L^* of L as a function of g and h.
- (b) Choose g so that L is formally self-adjoint.
- (c) Denoting the formally self-adjoint operator found in part (b) as \mathcal{L} , consider the Boundary Value Problem (BVP) on [0, 1/2] given by

$$\mathcal{L}u = 0,$$
 $au(0) + bu'(0) = 0, \ cu(1/2) + du'(1/2) = 0,$

with a, b, c, d some real nonzero constants. Is this BVP self-adjoint? Justify your answer fully by first giving the definition of a self-adjoint BVP and then checking that the given BVP satisfies the definition.





SECTION B

Q5 Let S be the "lampshade" surface given by the part of a cone with top rim $C_1 = \{x^2+y^2=1, z=1\}$ and bottom rim $C_0 = \{x^2+y^2=4, z=0\}$, and let $\boldsymbol{f}: \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$f(x) = 3yze_1 + x(z-1)e_2 + \sin(2025e^y)e_3.$$

- (a) Calculate the circulation of \boldsymbol{f} around C_0 if the curve is oriented anti-clockwise (looking from above).
- (b) Use an appropriate integral theorem to calculate the *outward* flux of $\nabla \times f$ through the surface S.
- Q6 A system of curvilinear coordinates is defined by

 $\boldsymbol{x}(u, v, w) = a \cosh u \cos v \boldsymbol{e}_1 + a \sinh u \sin v \boldsymbol{e}_2 + w \boldsymbol{e}_3,$

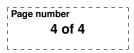
where a is a positive constant, $u \in [0, \infty)$, and $v \in [0, 2\pi]$.

- (a) Determine whether or not this coordinate system is orthogonal.
- (b) Show that the scale factor $h_u = a\sqrt{\sinh^2 u + \sin^2 v}$ and find h_v, h_w .
- (c) Let $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ denote the coordinate mapping from $(u, v, w) \to (x, y, z)$. Naming any theorem you use, determine at which points (x, y, z) this mapping has a differentiable inverse.
- **Q7** Consider the Sturm-Liouville eigenvalue problem on [1, b] for b > 1,

$$(x^2 u')' + \lambda u = 0,$$
 $u(1) = u(b) = 0.$

- (a) Identify the corresponding Sturm-Liouville operator \mathcal{L} and the weight ω .
- (b) Find the eigenvalues and the normalised eigenfunctions of the problem.

Hint: the substitution $x = e^t$ may be useful.



Q8 (a) Consider the two-dimensional domain D given by the intersection of the two half-planes

 $\Pi_{+} := \{ (x, y) \in \mathbb{R}^{2} : y \ge x \} \quad \text{and} \quad \Pi_{-} := \{ (x, y) \in \mathbb{R}^{2} : y \ge -x \},\$

i.e. $D = \Pi_+ \cap \Pi_-$.

Denote the origin of the plane \mathbb{R}^2 by O and label P the point in D with $OP := \mathbf{x_0} = \mathbf{e}_2$. Use the method of images to construct the Green's function $G(\mathbf{x}, \mathbf{x_0})$ satisfying

$$\nabla^2 G(\mathbf{x}, \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}) \qquad \text{for } \mathbf{x} \in D,$$

$$G(\mathbf{x}, \mathbf{x_0}) = 0 \qquad \text{for } \mathbf{x} \in \partial D,$$

where ∂D is the boundary of the domain D. You may use the fact that the fundamental solution of Laplace's equation, which is regular on \mathbb{R}^2 -{ $\mathbf{x_0}$ }, is given by

$$G_0(\mathbf{x}, \mathbf{x_0}) = \frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x_0}|.$$

Draw a rough sketch indicating the position of the point P and of its images to support your result for the Green's function $G(\mathbf{x}, \mathbf{x_0})$. Clearly mark the domain D, label your image points as P_i (with $\mathbf{OP_i} := \mathbf{x_i}$) and call Q the point such that $\mathbf{OQ} := \mathbf{x}$. Give your answer for the Green's function $G(\mathbf{x}, \mathbf{x_0})$ in terms of $\mathbf{x_0}, \mathbf{x}$ and $\mathbf{x_i}$.

- (b) Prove that the solution $G(\mathbf{x}, \mathbf{x_0})$ you obtained in part (a) satisfies $G(\mathbf{x}, \mathbf{x_0}) = 0$ everywhere on the boundary ∂D .
- (c) If one modifies the domain D such that the new domain is given by $D := \{(r,\theta) : 0 \le r \le 2, \pi/4 \le \theta \le 3\pi/4\}$, how many more images of the point P should one consider in order to construct the Green's function in this case? Draw a rough sketch with the positions of all the images of the point P of polar coordinates $(1, \pi/2)$ for the domain D and give the polar coordinates of the image point of P with respect to the boundary $\partial D_3 := \{(r, \theta) : r = 2, \pi/4 \le \theta \le 3\pi/4\}$.