Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2031-WE01-SP

Title: Analysis in Many Variables II (2023/24 Syllabus)

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Let $\boldsymbol{f}(\boldsymbol{x}) = z\cos(rz)\boldsymbol{e}_r + \big[r\cos(rz) + 1\big]\boldsymbol{e}_z$ in cylindrical coordinates.

(a) By finding a suitable potential $g(\boldsymbol{x})$, show that $\boldsymbol{f}$ is conservative – in other words, may be written $\boldsymbol{f} = \boldsymbol{\nabla} g$.

(b) Naming any theorem that you use, compute the line integral of $\boldsymbol{f}$ along an arbitrary curve from $(r, \theta, z) = (0, 0, 0)$ to $(r, \theta, z) = (3, \pi, \pi/2)$.

Q2 (a) Calculate the volume of the region $\{-1 \le z \le 1,\, r < 1 - z^2\}$ in cylindrical coordinates.

(b) Use index notation to simplify as much as possible the expression $\boldsymbol{\nabla} \cdot \big(\boldsymbol{x} \times \boldsymbol{a}\big)$, where $\boldsymbol{a}$ is a constant vector in $\mathbb{R}^3$.

Q3 Solve the following equation for the generalised function g,

$$(x - 1)(x^2 + x - 2)\, g(x) = 0,$$

i.e. find the generalised solution $g(x)$ in terms of shifted δ_a distributions and possibly their derivatives. Justify the steps taken to arrive at the solution.

Q4 You are given the linear operator

$$L = (1 - x^2)\,\frac{d^2}{dx^2} + g(x)\,\frac{d}{dx} + h(x),$$

with the two real-valued functions $g \in \mathcal{C}^1(\,(-1, 1)\,)$ and $h(x) \in \mathcal{C}^0(\,(-1, 1)\,)$.

(a) Calculate the formal adjoint L^* of L as a function of g and h.

(b) Choose g so that L is formally self-adjoint.

(c) Denoting the formally self-adjoint operator found in part (b) as $\mathcal{L}$, consider the Boundary Value Problem (BVP) on $[0, 1/2]$ given by

$$\mathcal{L}u = 0, \qquad au(0) + bu'(0) = 0,\; cu(1/2) + du'(1/2) = 0,$$

with a, b, c, d some real nonzero constants. Is this BVP self-adjoint? Justify your answer fully by first giving the definition of a self-adjoint BVP and then checking that the given BVP satisfies the definition.

SECTION B

Q5 Let S be the "lampshade" surface given by the part of a cone with top rim $C_1 = \{x^2+y^2 = 1,\ z = 1\}$ and bottom rim $C_0 = \{x^2+y^2 = 4,\ z = 0\}$, and let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$\boldsymbol{f}(\boldsymbol{x}) = 3yz\boldsymbol{e}_1 + x(z-1)\boldsymbol{e}_2 + \sin(2025e^y)\boldsymbol{e}_3.$$

(a) Calculate the circulation of $\boldsymbol{f}$ around C_0 if the curve is oriented anti-clockwise (looking from above).

(b) Use an appropriate integral theorem to calculate the *outward* flux of $\boldsymbol{\nabla} \times \boldsymbol{f}$ through the surface S.

Q6 A system of curvilinear coordinates is defined by

$$\boldsymbol{x}(u, v, w) = a\cosh u \cos v\boldsymbol{e}_1 + a\sinh u \sin v\boldsymbol{e}_2 + w\boldsymbol{e}_3,$$

where a is a positive constant, $u \in [0, \infty)$, and $v \in [0, 2\pi]$.

(a) Determine whether or not this coordinate system is orthogonal.

(b) Show that the scale factor $h_u = a\sqrt{\sinh^2 u + \sin^2 v}$ and find h_v, h_w.

(c) Let $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ denote the coordinate mapping from $(u, v, w) \to (x, y, z)$. Naming any theorem you use, determine at which points (x, y, z) this mapping has a differentiable inverse.

Q7 Consider the Sturm-Liouville eigenvalue problem on $[1, b]$ for $b > 1$,

$$(x^2\, u')' + \lambda u = 0, \qquad u(1) = u(b) = 0.$$

(a) Identify the corresponding Sturm-Liouville operator $\mathcal{L}$ and the weight ω.

(b) Find the eigenvalues and the normalised eigenfunctions of the problem.

Hint: the substitution $x = e^t$ may be useful.

Q8 (a) Consider the two-dimensional domain D given by the intersection of the two half-planes

$$\Pi_+ := \{(x,y) \in \mathbb{R}^2 : y \geq x\} \qquad \text{and} \qquad \Pi_- := \{(x,y) \in \mathbb{R}^2 : y \geq -x\},$$

i.e. $D = \Pi_+ \cap \Pi_-$.

Denote the origin of the plane $\mathbb{R}^2$ by O and label P the point in D with $\mathbf{OP} := \mathbf{x_0} = \mathbf{e}_2$. Use the method of images to construct the Green's function $G(\mathbf{x}, \mathbf{x_0})$ satisfying

$$\begin{aligned} \nabla^2 G(\mathbf{x}, \mathbf{x_0}) &= \delta(\mathbf{x} - \mathbf{x_0}) && \text{for } \mathbf{x} \in D, \\ G(\mathbf{x}, \mathbf{x_0}) &= 0 && \text{for } \mathbf{x} \in \partial D, \end{aligned}$$

where ∂D is the boundary of the domain D. You may use the fact that the fundamental solution of Laplace's equation, which is regular on $\mathbb{R}^2$-$\{\mathbf{x_0}\}$, is given by

$$G_0(\mathbf{x}, \mathbf{x_0}) = \frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x_0}|.$$

Draw a rough sketch indicating the position of the point P and of its images to support your result for the Green's function $G(\mathbf{x}, \mathbf{x_0})$. Clearly mark the domain D, label your image points as P_i (with $\mathbf{OP_i} := \mathbf{x_i}$) and call Q the point such that $\mathbf{OQ} := \mathbf{x}$. Give your answer for the Green's function $G(\mathbf{x}, \mathbf{x_0})$ in terms of $\mathbf{x_0}, \mathbf{x}$ and $\mathbf{x_i}$.

(b) Prove that the solution $G(\mathbf{x}, \mathbf{x_0})$ you obtained in part (a) satisfies $G(\mathbf{x}, \mathbf{x_0}) = 0$ everywhere on the boundary ∂D.

(c) If one modifies the domain D such that the new domain is given by $\widetilde{D} := \{(r, \theta) : 0 \leq r \leq 2, \pi/4 \leq \theta \leq 3\pi/4\}$, how many more images of the point P should one consider in order to construct the Green's function in this case? Draw a rough sketch with the positions of all the images of the point P of polar coordinates $(1, \pi/2)$ for the domain $\widetilde{D}$ and give the polar coordinates of the image point of P with respect to the boundary $\partial D_3 := \{(r, \theta) : r = 2, \pi/4 \leq \theta \leq 3\pi/4\}$.