Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2031-WE01

Title:

Analysis in Many Variables II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Let $\boldsymbol{f}(\boldsymbol{x}) = z\cos(rz)\boldsymbol{e}_r + \left[r\cos(rz) + 1\right]\boldsymbol{e}_z$ in cylindrical coordinates.

(a) By finding a suitable potential $g(\boldsymbol{x})$, show that $\boldsymbol{f}$ is conservative – in other words, may be written $\boldsymbol{f} = \boldsymbol{\nabla} g$.

(b) Naming any theorem that you use, compute the line integral of $\boldsymbol{f}$ along an arbitrary curve from $(r, \theta, z) = (0, 0, 0)$ to $(r, \theta, z) = (3, \pi, \pi/2)$.

Q2 (a) Calculate the volume of the region $\{-1 \leq z \leq 1,\, r < 1 - z^2\}$ in cylindrical coordinates.

(b) Use index notation to simplify as much as possible the expression $\boldsymbol{\nabla} \cdot \left(\boldsymbol{x} \times \boldsymbol{a}\right)$, where $\boldsymbol{a}$ is a constant vector in $\mathbb{R}^3$.

Q3 Consider the function

$$u(x,t) = \begin{cases} \dfrac{3\left(1 - \dfrac{x^2}{t}\right)}{4\sqrt{t}} & |x| \leq t^{1/2}, \\ 0 & |x| > t^{1/2}. \end{cases}$$

Show that it can be used to define a delta distribution in the limit $t \to 0$. You may use any results stated in your class notes.

Q4 Find the Green's function $G(x, \xi)$ for the operator:

$$L = \frac{d^2}{dx^2} - \frac{d}{dx},$$

on $x \in [0, 1]$, subject to the boundary conditions

$$G(0, \xi) = 0, \quad \frac{\partial G}{\partial x}(1, \xi) = \mathrm{e}^1 = \mathrm{e}.$$

SECTION B

Q5 Let S be the "lampshade" surface given by the part of a cone with top rim $C_1 = \{x^2+y^2 = 1,\ z = 1\}$ and bottom rim $C_0 = \{x^2+y^2 = 4,\ z = 0\}$, and let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$\boldsymbol{f}(\boldsymbol{x}) = 3yz\boldsymbol{e}_1 + x(z-1)\boldsymbol{e}_2 + \sin(2025e^y)\boldsymbol{e}_3.$$

(a) Calculate the circulation of $\boldsymbol{f}$ around C_0 if the curve is oriented anti-clockwise (looking from above).

(b) Use an appropriate integral theorem to calculate the *outward* flux of $\boldsymbol{\nabla} \times \boldsymbol{f}$ through the surface S.

Q6 A system of curvilinear coordinates is defined by

$$\boldsymbol{x}(u, v, w) = a \cosh u \cos v \boldsymbol{e}_1 + a \sinh u \sin v \boldsymbol{e}_2 + w\boldsymbol{e}_3,$$

where a is a positive constant, $u \in [0, \infty)$, and $v \in [0, 2\pi]$.

(a) Determine whether or not this coordinate system is orthogonal.

(b) Show that the scale factor $h_u = a\sqrt{\sinh^2 u + \sin^2 v}$ and find h_v, h_w.

(c) Let $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ denote the coordinate mapping from $(u, v, w) \to (x, y, z)$. Naming any theorem you use, determine at which points (x, y, z) this mapping is a local diffeomorphism.

Q7 Consider a domain $D \in \mathbb{R}^2$ which has the shape of an arrow head. Its coordinates (r, s) are given by the functions

$$x_1(r, s) = rs \quad \text{and} \quad x_2(r, s) = \frac{1}{2}(r^2 - s^2),$$

where x_1 and x_2 are points in Cartesian space and $r \in [0.1, 1]$ and $s \in [0.1, 1]$ are the coordinates which parametrise this domain.

In this coordinate system the Laplacian takes the form

$$\nabla^2\psi = \frac{1}{r^2 + s^2}\left(\frac{\partial^2\psi}{\partial s^2} + \frac{\partial^2\psi}{\partial r^2}\right).$$

(a) Consider the eigenvalue problem

$$\nabla^2\psi + k^2\psi = 0. \tag{1}$$

By assuming a solution which is separable in r and s, $\psi(r, s) = \phi(r)\xi(s)$, show that the eigen-equation (1) can be reduced to the following two ODE's

$$\frac{1}{\xi}\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + k^2 s^2 = m, \quad \frac{1}{\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + k^2 r^2 = -m,$$

with m some arbitrary real number.

(b) Derive the general form for the solutions ψ of (1) in the $k = 0$ case.

(c) Consider the following problem on this domain:

$$\nabla^2\psi = f(r, s),$$

paired with homogeneous Neumann boundary conditions. Show that the Fredholm alternative implies there can be no unique solution to this problem.

Q8 Consider the model for a density $u(x, t) : [0, L] \times [0, \infty] \to \mathbb{R}$ given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = Lu, \quad L = \frac{\partial^2}{\partial x^2} - \gamma,$$

subject to the boundary conditions $\partial u/\partial x|_{x=0} = \partial u/\partial x|_{x=L} = 0$. Here, γ is a real-valued constant.

(a) Find the general solution as an eigenfunction expansion in the spatial variable of the form

$$u(x, t) = \sum_{n=0}^{\infty} c_n(t) y_n(x).$$

(b) Each mode growth function $c_n(t)$ has a value given by an initial condition $c_n(0) = 0$ and $\mathrm{d}c_n/dt|_{t=0} < 0$. State which modes can satisfy the condition $c_n(t_p) > c_n(0)$ for $t_p > 0$ and the range of γ values for which this is the case.