EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2051-WE01

Title:
Numerical Analysis II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 (a) Define what is meant by a floating-point number having a *finite precision* and a *finite range*.

(b) Explain why in Python `1.0 + 2.0 - 3.0` gives `0.0` but `0.1 + 0.2 - 0.3` does not.

(c) Let the floating-point numbers x and y have relative errors δ_x and δ_y. Show that the relative error δ in their product satisfies

$$|1+\delta| \leq \exp(|\delta_x| + |\delta_y|).$$

You may assume that $xy \neq 0$ and $|\delta_x|$, $|\delta_y| < 1$.

Q2 (a) Assuming smooth f, derive the error of the forward difference formula

$$\mathcal{E} = f'(x) - \frac{f(x+h) - f(x)}{h}$$

and use it to show that forward difference is first-order in accuracy.

(b) Now derive the error when the approximation is computed using floating-point arithmetic,

$$\mathcal{E}' = f'(x) - \frac{f(x+h) \ominus f(x)}{h}.$$

For simplicity, assume that all operations except the $\ominus$ are exact.

(c) From your $\mathcal{E}'$, explain the role of round-off error in this approximation.

Q3 (a) Define what is meant by the 1-norm for a vector, an induced norm (for a matrix), and the column-sum norm for a matrix.

(b) Considering 2×2 real matrices *only*, prove that the column-sum norm is the norm induced by the vector 1-norm.

Q4 Compute polynomials $\{H_n\}_{n=0}^{3}$ with $H_n \in \mathcal{P}_n$ orthogonal in the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x.$$

The following relations may be useful:

$$\int_{-\infty}^{\infty} x^n \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{n-1}{2}\int_{-\infty}^{\infty} x^{n-2}\mathrm{e}^{-x^2}\,\mathrm{d}x.$$

SECTION B

Q5 (a) State (do not prove) the contraction mapping theorem.

(b) Using the contraction mapping theorem, prove the local convergence theorem: if $g \in C^1$ has a fixed point x_* with $|g'(x_*)| < 1$, then the iteration $x_{n+1} = g(x_n)$ converges to x_* for x_0 sufficiently close to x_*.

(c) Assuming that $x = \sin(2x)$ has a unique positive solution x_*, does the iteration

$$x_{n+1} = \sin(2x_n)$$

converge to x_*? And if so, is the convergence monotone? Justify your answer.

Q6 We seek a $p \in \mathcal{P}_3$ such that $p(\pm 1) = 0$, $p(0) = 5$ and $p'(0) = 0$.

(a) Compute the divided difference table using the Newton interpolation construction, with $[a, a]f := \lim_{y \to a}[a, y]f = f'(a)$, and write down $p(x)$.

(b) Determine whether the resulting p is unique. Justify your answer.

(c) Assuming that the given data arise from a smooth function f, show that one has the error formula

$$f(x) - p(x) = \frac{(x^2 - 1)x^2}{4!} f^{(iv)}(\xi)$$

for some $\xi(x) \in [-1, 1]$.

Q7 (a) Perform LU decomposition on

$$A = \begin{pmatrix} 2 & 7 & 1 & 8 \\ 6 & 23 & 11 & 25 \\ 2 & 15 & 41 & 14 \\ 2 & 17 & 113 & 31 \end{pmatrix}.$$

(b) Solve $Ax = b = (89, 293, 225, 507)^{\mathrm{T}}$ given that $x_4 = 8$.

(c) Which singular matrices can be written as $A = LU$, where U is upper triangular and L lower triangular with $L_{jj} = 1$: all, some or none? Justify your answer.

Q8 (a) Define what is meant by a closed Newton–Cotes formula.

(b) Show that the $n = 3$ closed Newton–Cotes formula in the interval $[a, b]$ is

$$\mathcal{I}_3(f) = \frac{b-a}{8}\Big[f(a) + 3f\Big(\frac{2a+b}{3}\Big) + 3f\Big(\frac{a+2b}{3}\Big) + f(b)\Big].$$

(c) One seeks to construct a four-node quadrature formula in $[-1, 1]$ that must include $x_0 = -1$ and $x_3 = 1$. Determine the remaining nodes $\{x_1, x_2\}$ and the coefficients $\{\rho_0, \cdots, \rho_3\}$ that give degree of exactness of (at least) 5.