Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2025	MATH2071-WE01

Title: Mathematical Physics II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Write your answer in the white-covered answer booklet with barcodes. Begin your answer to each question on a new page.

Revision:	

ED01/2025

SECTION A

Q1 Consider a Lagrangian of the form

$$L = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}mq_1^4\dot{q}_2^2 - \frac{1}{6}q_1^6\,,$$

with m a constant and q_1, q_2 generalised coordinates.

(a) Find the Euler-Lagrange equations of motion for q_1 and q_2. You do not need to solve them.

(b) There is an ignorable coordinate in this system. Identify it, and write down the associated conserved charge.

(c) Verify, by computing the time derivative and using the Euler-Lagrange equations of motion, that the conserved charge you found in the previous part is indeed conserved.

Q2 Close to equilibrium, a system is described by the approximate Lagrangian

$$L_{\text{approx}} = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 - \frac{1}{2}(9q_1^2 + 4q_1q_2 + 6q_2^2)\,.$$

Find the general solution for $q_1(t)$ and $q_2(t)$.

Q3 A particle confined to a ring of circumference L has basis wave functions at $t = 0$ given by

$$\psi_n(t = 0, x) = \frac{1}{\sqrt{L}}e^{2\pi inx/L}\,, \quad n \in \mathbb{Z}\,. \tag{1}$$

(a) Are these eigenfunctions of the position operator? Of the momentum operator? Of the Hamiltonian? Motivate your answer in each case.

(b) How do these wave functions evolve under time evolution?

(c) Compute the expectation value $\langle p \rangle$ when the system is in the state given by ψ_n.

Q4 Consider a quantum particle on the real line. We are going to use the position representation, in which the position and momentum operators are represented by

$$\hat{x} = x\,, \quad \hat{p} = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\,. \tag{2}$$

(a) Use the inner product between two wave functions $\psi(x)$ and $\phi(x)$ to give the definition of the Hermitian conjugate of an operator $\hat{Q}(\hat{x}, \hat{p})$.

(b) Now define the following composite operators,

$$\hat{A} = \hat{x}\hat{p} + \hat{p}\hat{x}\,, \quad \hat{B} = \hat{x}^2\hat{p}\,. \tag{3}$$

Show that $\hat{A}$ is Hermitian by using the definition of Hermitian conjugate you gave above.

(c) Show that $[\hat{x}, \hat{p}] = i\hbar$ and then use this to compute the commutator $[\hat{A}, \hat{B}]$. Express your answer in terms of $\hat{x}$ and $\hat{p}$.

SECTION B

Q5 Consider a string described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho(x)u_t^2 - \frac{1}{2}\tau u_x^2$$

where $u(x,t)$ is a field and $u_x \coloneqq \partial u/\partial x$ and $u_t \coloneqq \partial u/\partial t$ are its space and time derivatives. We take τ to be constant, but

$$\rho(x) = \begin{cases} \rho_0 & \text{when } x < 0\,, \\ 4\rho_0 & \text{when } x \geq 0\,, \end{cases}$$

with ρ_0 a constant.

(a) Write down the Euler-Lagrange equation of motion valid away from $x = 0$, and give its general solution.

(b) Recall that the energy-momentum tensor is given by

$$T_{ij} = u_i \frac{\partial \mathcal{L}}{\partial u_j} - \delta_{ij}\mathcal{L}\,.$$

In particular T_{tx} encodes the energy flux. Write down the explicit equation for energy conservation at $x = 0$ for the Lagrangian density $\mathcal{L}$ given above.

(c) Fix (in terms of τ, ρ_0 and p) the constants c, $\tilde{c}$, R and T in the following ansatz so that $u(x,t)$ satisfies the equations of motion away from $x = 0$, is continuous at $x = 0$, and conserves energy at $x = 0$:

$$u(x,t) = \begin{cases} \mathrm{Re}\left(e^{-ipct}(e^{ipx} + Re^{-ipx})\right) & \text{for } x < 0\,, \\ \mathrm{Re}\left(Te^{-2ip\tilde{c}t}e^{2ipx}\right) & \text{for } x \geq 0\,, \end{cases}$$

where $\mathrm{Re}(\cdot)$ denotes taking the real part.

Q6 The energy associated to a Lagrangian L is given by

$$E = \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L\,.$$

(a) Assume that L has no explicit dependence on time, and only depends on time implicitly through its dependence on $\mathbf{q}$ and $\dot{\mathbf{q}}$. That is, $L = L(\mathbf{q}, \dot{\mathbf{q}})$. Show, by explicitly taking the time derivative and using the Euler-Lagrange equations of motion, that

$$\frac{dE}{dt} = 0\,.$$

(b) Consider, as an example, the Lagrangian

$$L = \frac{1}{6}(\dot{q}_1^2 + \dot{q}_2^2)^3 - \frac{1}{6}(q_1^2 + q_2^2)^3\,.$$

(Note the extra cube powers in both terms.) Compute the energy associated to this Lagrangian.

(c) Construct the Hamiltonian H for the system in the previous part, as a function of the positions and generalised momenta.

(d) Consider rotations around the origin in the (q_1, q_2) plane:

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\,.$$

Construct the Noether charge

$$Q = \sum_i a_i \frac{\partial L}{\partial \dot{q}_i}$$

associated to this transformation (where a_i is the generator of the transformation of q_i), and compute the Poisson bracket $\{Q, H\}$.

Q7 Consider a system of two non-interacting particles A and B, each free to move in one dimension. At $t = 0$ the system is prepared in the following state:

$$\psi(x_a, x_b, t = 0) = \frac{1}{\sqrt{2}} \left(e^{ikx_a} e^{2ikx_b} + e^{2ikx_a} e^{ikx_b} \right)\,, \tag{4}$$

where k is some positive constant.

(a) Is this state entangled? Justify your answer.

(b) If you measure the momentum of particle A at $t = 0$, what are the possible outcomes and their corresponding probabilities?

(c) Immediately after measuring particle A's momentum, what can you say with certainty about particle B's momentum?

(d) Write down the time evolution of this state for $t > 0$. Is the entanglement affected by time evolution?

Q8 A quantum particle is subject to a harmonic oscillator potential with time-dependent frequency,

$$V(x,t) = \frac{1}{2} m\omega(t)^2\, x^2\,. \tag{5}$$

The frequency $\omega(t)$ is a simple step as a function of time,

$$\omega(t) = \begin{cases} \omega & \text{for } t < 0\,, \\ \omega' & \text{for } t \geq 0\,, \end{cases} \tag{6}$$

where ω and ω' are constants. At some time $t < 0$ the system is described by the wave function

$$\psi(x, t = t_0 < 0) = \phi_0(x)\,, \tag{7}$$

where ϕ_0 is the ground state wave function of the harmonic oscillator with frequency ω.

(a) What does the time-dependence of the wave function for $t < 0$ look like?

(b) If a measurement is made of the energy at any time $t < 0$, what is/are the possible outcome(s)?

(c) Immediately after the frequency change at $t = 0$, we can assume that the wave function is still unchanged. If $\omega' = 3\omega$, explain why the probability of measuring energy $\frac{1}{2}\hbar\omega$ at $t > 0$ is zero.

(d) What is the probability of measuring energy $\frac{1}{2}\hbar\omega'$ just after $t = 0$, expressed as a function of ω and ω'? Discuss the limiting situations $\omega' = \omega$ and $\omega' \to \infty$. [You may use without proof that $\int_{-\infty}^{\infty} \exp(-ax^2)\mathrm{d}x = \sqrt{\pi/a}$.]